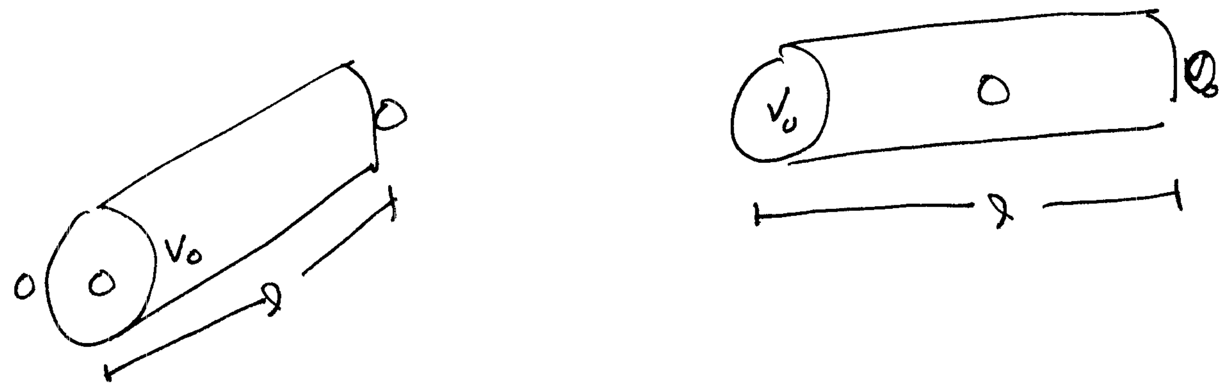
Bessel Functions

Let's relax the restriction that z is uniform.

Consider the systems



Laplace Eqn Cylindrical Coordinates

$$\frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Separated Solution

$$V(\rho, \phi, z) = P(\rho)\Phi(\phi)Z(z) = R(\rho, \phi)Z(z)$$

Separate $Z(z)$ first to give

$$\frac{1}{R}\left(\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 R}{\partial \phi^2}\right) + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0$$

$$Z = \cancel{e^{\pm ikz} \text{ or } \sin kz, \cos kz}$$

$$e^{\pm kx}$$

Now we have

$$\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 R}{\partial \phi^2} + k^2 R = 0$$

$$\rho^2 \frac{\partial^2 R}{\partial \rho^2} + \rho \frac{\partial R}{\partial \rho} + \rho^2 k^2 R + \frac{\partial^2 R}{\partial \phi^2} = 0$$

<u>Seprate Again</u>

$$R = P(\rho)\,\Phi(\phi)$$

$$\underbrace{\frac{1}{P}\left[\rho^2 \frac{\partial^2 P}{\partial \rho^2} + \rho \frac{\partial P}{\partial \rho} + \rho^2 k^2 P\right]}_{+\nu^2} + \underbrace{\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2}}_{-\nu^2} = 0$$

where ν integer

$$\frac{d^2 \Phi}{d\phi^2} + \nu^2 \Phi = 0 \qquad \Phi = \begin{matrix} \sin \nu\phi \\ \cos \nu\phi \end{matrix}$$

Leaving

$$\rho^2 \frac{\partial^2 P}{\partial \rho^2} + \rho \frac{\partial P}{\partial \rho} + \rho^2 k^2 P = \nu^2 P$$

$$\frac{\partial^2 P}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial P}{\partial \rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right) P = 0$$

Let $\quad x = \dfrac{x_{0n}\,\rho}{a} \qquad dx = \dfrac{x_{0n}}{a}\,d\rho$

$$\rho = \frac{a}{x_{0n}}$$

$$A_n = \frac{2V_0}{a^2 J_1^2(x_{0n}) \sinh(k_{0n}\ell)} \left(\frac{a}{x_{0n}}\right)^2 \int_0^{x_{0n}} x\, J_0(x)\, dx$$

$$\int_0^{x_{0n}} x\, J_0(x)\, dx = x\, J_1(x)\Big|_0^{x_{0n}} = x_{0n} J_1(x_{0n})$$

$$A_n = \frac{2V_0}{x_{0n} J_1(x_{0n}) \sinh(k_{0n}\ell)}$$

Let $x = k\rho$

$$\frac{d^2P}{dx^2} + \frac{1}{x}\frac{dP}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)P = 0$$

$\Rightarrow$ Bessel's Eqn

<u>Solutions Bessel Functions</u>

~~$J_m(x)$~~ $J_\nu(x)$

<u>Neumann Functions</u>

~~#~~ $N_\nu(x)$

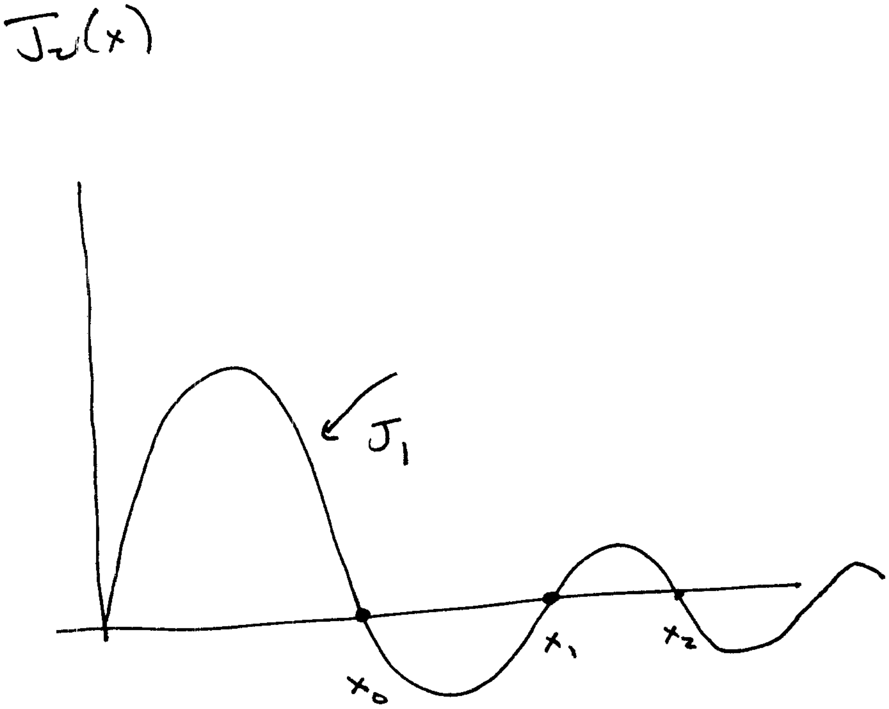
Gamma Function

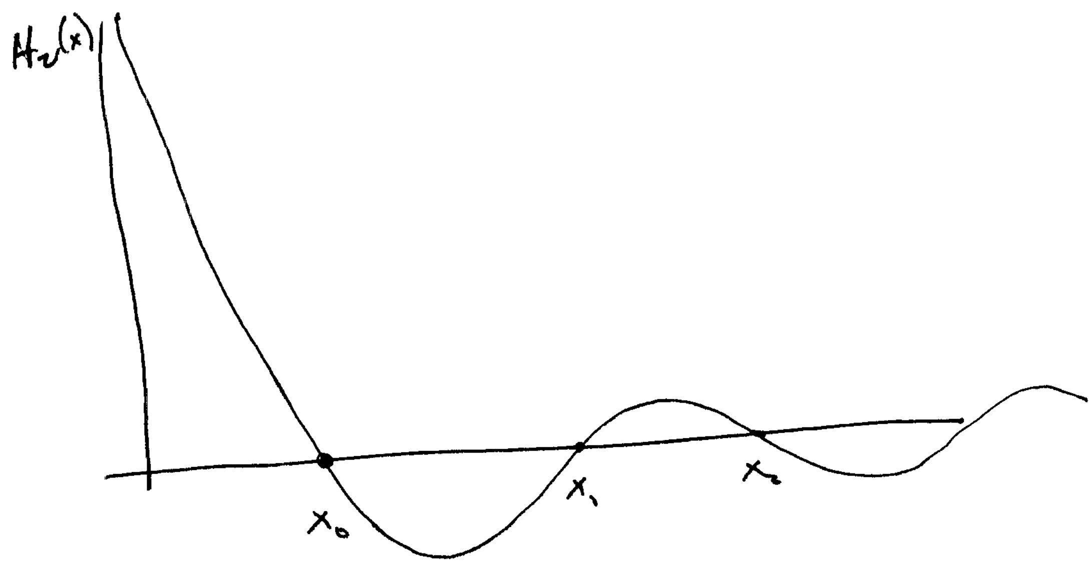
$\Gamma(x) = (x-1)!$

<u>Limits</u>

$$J_\nu(x) = \begin{cases} \dfrac{1}{\Gamma(\nu+1)}\left(\dfrac{x}{2}\right)^\nu & x \ll 1 \\ \sqrt{\dfrac{2}{\pi x}}\cos\left(x - \dfrac{\nu\pi}{2} - \dfrac{\pi}{4}\right) & x \gg 1 \end{cases}$$

$$N_\nu(x) = \begin{cases} \dfrac{2}{\pi}\left[\ln\left(\dfrac{x}{2}\right) + 0.5772 + \cdots\right] & \nu = 0 \quad x \ll 1 \\ -\dfrac{\Gamma(\nu)}{\pi}\left(\dfrac{2}{x}\right)^\nu & \nu \neq 0 \quad x \ll 1 \\ \sqrt{\dfrac{2}{\pi x}}\sin\left(x - \dfrac{\nu\pi}{2} - \dfrac{\pi}{4}\right) & x \gg 1 \end{cases}$$





Each Bessel function has a set of zero

$\nu = 0 \quad x_{01} = 2.405 \qquad x_{02} = 5.520 \cdots$

$\nu = 1 \quad x_{11} = 3.832 \qquad x_{12} = 7.016 \cdots$

$$J_\nu(x_{\nu n}) = 0$$

Orthogonality

$$\int_0^a \rho\, J_\nu\!\left(x_{\nu m} \frac{\rho}{a}\right) J_\nu\!\left(x_{\nu n} \frac{\rho}{a}\right) d\rho$$

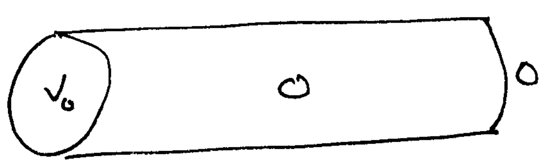
$$= \frac{a^2}{2} \left[J_{\nu+1}\left(x_{\nu n}\right)\right]^2 \delta_{mn}$$

Completeness

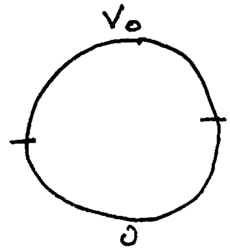
$$f(\rho) = \sum_{n=1}^{\infty} A_{\nu n} J_\nu\!\left(x_{\nu n}\, \rho/a\right)$$

$$A_{\nu n} = \frac{2}{a^2 J_{\nu+1}^2(x_{\nu n})} \int_0^a \rho\, f(\rho)\, J_\nu\!\left(\frac{x_{\nu n}\rho}{a}\right) d\rho$$

So these can be used to solve



The other case we considered is



of length l with both ends zero.

This time the Z separation must yield sines and cosines.

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -k^2$$

Gives
$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - \left(1 + \frac{\nu^2}{x^2}\right)R = 0$$

$\Rightarrow$ Solutions Modified Bessel Functions

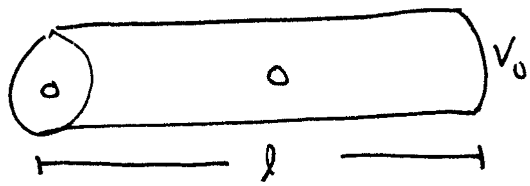
$$I_\nu(x) = i^{-\nu} J_\nu(ix)$$

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix)$$

Hankel Function

$$H_\nu^{(1)}(x) = J_\nu(x) + i N_\nu(x)$$

For our cylinder,



our possible solutions are $1, \ln\rho, \phi, z$

$$\left[J_\nu(\rho k) + N_\nu(\rho k)\right] \times \left[\sin\nu\phi, \cos\nu\phi\right]\left[e^{kz}, e^{-kz}\right]$$

All the trivial solutions can be eliminated. N_ν blows up.

To satisfy $V=0$ at $z=0$ take $\dfrac{e^{kz}-e^{-kz}}{2} = \sinh kz$

for the z dependance.

There is no ϕ dependance, so used $\nu=0$ solution. This

leaves us

$J_0(\rho k) \sinh kz$ for our solutions

Now satisfy the boundary condition $V(a,\phi)=0$.

$$\Rightarrow \quad J_0(ak) = 0$$

$$\Rightarrow \quad k_{0n} = \frac{x_{0n}}{a}$$ where x_{mn} are the zeros of the Bessel functions.

The complete solution is then

$$V(\rho,\phi,z) = \sum_n A_n J_0(x_{0n}\rho/a)\sinh(k_{0n} z)$$

Satisfy final boundary condition

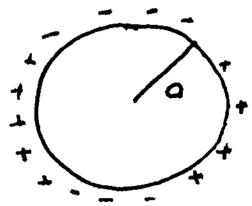
$$V(\rho,\phi,l) = V_0 = \sum_n A_n J_0(x_{0n}\rho/a)\sinh(k_{0n} l)$$

$$A_n = \frac{2V_0}{a^2 J_1^2(x_{0n})\sinh(k_{0n}l)}\int_0^a \rho\, J_0\!\left(\frac{x_{0n}\rho}{a}\right)d\rho$$

$$\cancel{= \frac{2V_0}{a^2 J_1^2(x_{0n})\sinh(k_{0n}l)}}$$

Ex An infinitely long cylinder has a potential at the surface of $V(a,\phi) = V_0 \cos 2\phi$

Compute field everywhere.



Solutions $1,\ \ln \rho,\ \phi$

$\rho^n \cos n\phi$ $\qquad$ $\rho^{-n} \cos n\phi$

$\rho^n \sin n\phi$ $\qquad$ $\rho^{-n} \sin n\phi$

Inside

$$V(\rho,\phi) = \sum_n A_n \rho^n \cos n\phi + B_n \rho^n \sin n\phi$$

$$V(a,\phi) = V_0 \cos 2\phi = \sum A_n a^n \cos n\phi + B_n a^n \sin n\phi$$

$$\Longrightarrow \quad A_n,\ B_n = 0 \quad \text{except} \quad A_2 = \frac{V_0}{a^2}$$

$$V(\rho,\phi)_{inside} = \frac{V_0}{a^2}\rho^2 \cos 2\phi$$

<u>Outside</u>

$$V(\rho,\phi) = \sum A_n \rho^{-n} \cos n\phi + B_n \rho^{-n} \sin n\phi$$

$$V(a,\phi) = V_0 \cos 2\phi$$

$A_n, B_n = 0$ except

$$A_2 = a^2 V_0$$

$$V(\rho,\phi) = \frac{V_0 a^2}{\rho^2} \cos 2\phi$$

Field Inside

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi}$$

$$\vec{E}_{inside} = -\nabla V$$

$$= \frac{-2V_0}{a^2}\rho\cos 2\phi\,\hat{\rho} + \frac{2V_0}{a^2}\rho\sin 2\phi\,\hat{\phi}$$

$$\vec{E}_{outside} = \frac{2V_0 a^2}{\rho^3}\cos 2\phi\,\hat{\rho} + \frac{2V_0 a^2}{\rho^3}\sin 2\phi\,\hat{\phi}$$

Surface Charge Use Pill box at surface

$$\sigma = \left(\vec{E}_{outside} - E_{inside}\right)\cdot\hat{\rho}\,\varepsilon_0$$

$$\sigma = \frac{2V_0}{a}\cos 2\phi - \left(-\frac{2V_0}{a}\cos 2\phi\right)$$

$$= \frac{4V_0}{a}\cos 2\phi$$