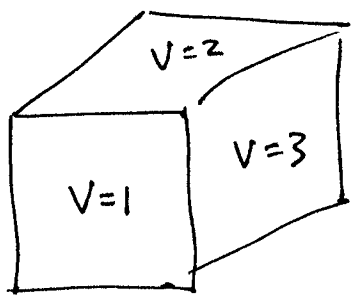
Boundary Value Problem

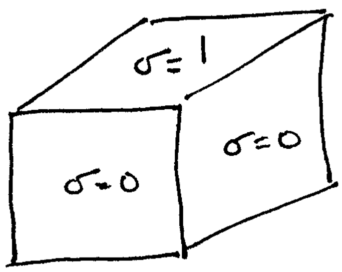
(1) Specify charge/current distribution in the system (fixed).

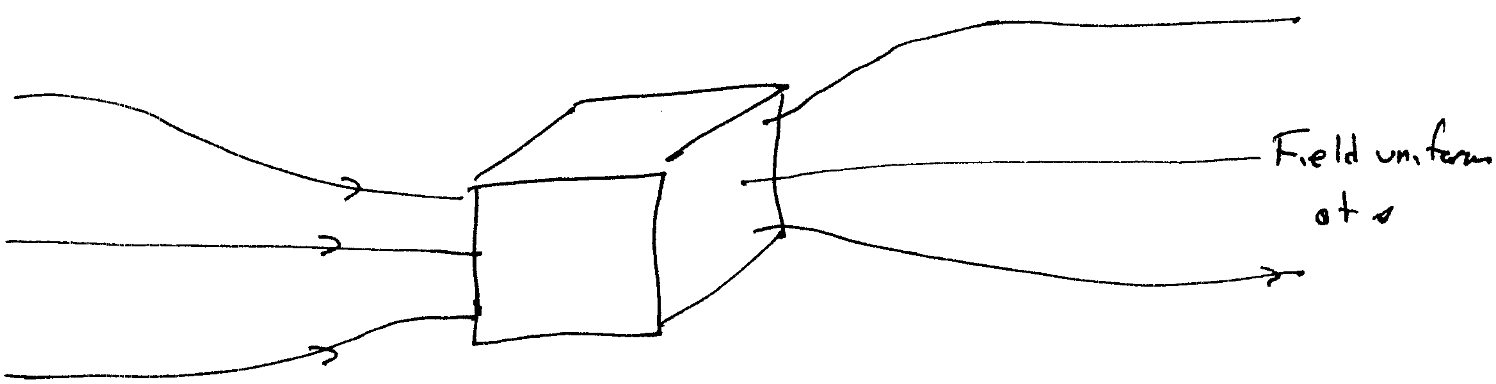
(2) Specify V or $\hat{n}\cdot\nabla V$ on the boundary.

$\Rightarrow$ Boundary can be ∞

Examples







For charge $\left(\frac{\partial \rho}{\partial t} = 0\right)$

Poisson's Equation

$$\nabla^2 V = -\rho/\varepsilon_0$$

Laplace's Eqn (charge free region - charge on boundaries)

$$\nabla^2 V = 0$$

Magnetic Analogue (No free current)

$$\nabla \times \vec{H} = 0$$

$$\vec{H} = -\nabla V_m \qquad \text{magnetic scaler potential}$$

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M}$$

$$\nabla \cdot \vec{B} = \mu_0 \nabla \cdot \vec{H} + \mu_0 \nabla \cdot \vec{M}$$

$$\nabla \cdot \vec{H} = -\nabla \cdot \vec{M}$$

$$\boxed{\nabla^2 V_m = \nabla \cdot \vec{M}}$$

Homogeneous and Inhomogeneous Equations -

If f_a and f_b are solutions to some differential equation satisfying some boundary conditions, then the equation is homogeneous if $f_a + f_b$ is also a solution. Laplace's eqn is homogeneous, Poisson's equation is not.

When presented with an inhomogeneous equation, we find ANY solution to the equation V_p (the particular solution) and then fix the boundary conditions with the homogeneous part of the solution.

So given $\nabla^2 V = -\rho/\epsilon_0$ find any V_p that solves the equation $\nabla^2 V_p = -\rho/\epsilon_0$ then find V_h where $\nabla^2 V_h = 0$ to satisfy the boundary conditions. The total solution is then $V = V_h + V_p$

$$\frac{\partial V}{\partial x^2} + \frac{\partial V}{\partial y^2} = 0$$

Separation of Variables

Propose $V = X(x)\,Y(y)$

- Does not have to give a solution
- Does not give all solutions

$$Y(y)\frac{\partial^2 X}{\partial x^2} + X(x)\frac{\partial^2 Y}{\partial y^2} = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} \equiv K^2$$

Since the equations vary independantly.

Note, we can solve the pieces independanty

with 1, x, y.

Generally, the particular solution has to be found by intuition, trickery, or luck.

Let's focus on solving the homogeneous problem

Laplace's Eqn $\quad \nabla^2 V = 0$

Cartesian ~~Rectangular~~ Coordinates

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial V^2}{\partial z^2} = 0$$

1D $\quad - \quad V(x,y,z) = V(x)$

$$\nabla^2 V \Rightarrow \frac{d^2 V}{dx^2} = 0$$

$$V = V_0 + V_1 x$$

2D $\quad V(x,y,z) = V(x,y)$

The x and y component then separate

$$\frac{d^2X}{dx^2} = k^2 X \qquad \frac{d^2Y}{dy^2} = -k^2 Y$$

Trivial Solutions $k^2=0$

$X=1$, $X=x$ $\qquad$ $Y=1$, $Y=y$

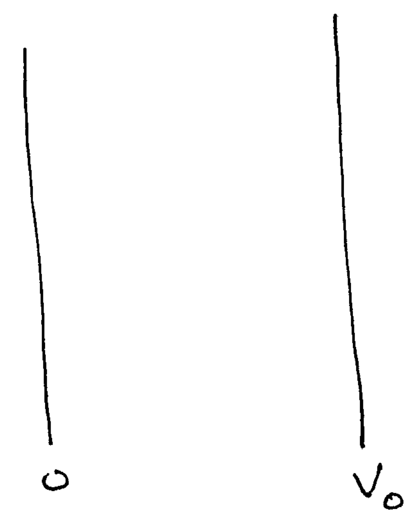
Non-trivial solutions	
$X = Ae^{kx}$	$Y = A\sin kx$
$X = Ae^{-kx}$	$Y = A\cos kx$

But what is k? It will be fixed by the boundary conditions.

Let's solve a simple system to get the feel of it.

An infinite plate in the y-z plane is held at zero potential. A parallel plane through the point $x=d$ is held at V_0. Compute the field between the plates.

There is no charge between the plates, so Laplace's eqns determines the potential. The problem is one-dimensional.



$$\frac{d^2V}{dx^2} = 0$$

solutions $1, x$

$$V(x) = A + Bx$$

$$V(0) = A = 0$$

$$V(d) = Bd = V_0 \qquad B = \frac{V_0}{d}$$

$$V(x) = \frac{V_0}{d} x$$

Now find field and charge

$$\vec{E} = -\frac{dV}{dx}\hat{x} = -\frac{V_0}{d}\hat{x}$$

Charge on left plane

$$EA - 0 = \frac{\sigma A}{\varepsilon_0}$$ (Gaussian pillbox)

$$-\frac{V_0}{d} = \frac{\sigma_l}{\varepsilon_0}$$

$$\sigma_l = -\varepsilon_0 \frac{V_0}{d} = -\sigma_r$$

The full solution is then

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n\ odd} \frac{1}{n(1-e^{-2k_n b})} \left(e^{-k_n x} - e^{+k_n(x-2b)}\right) \sin k_n y$$

The electric field in the cavity is

$$\vec{E} = -\nabla V$$

and the charge densities on the wall

$$\varepsilon_0 E_n = \sigma$$

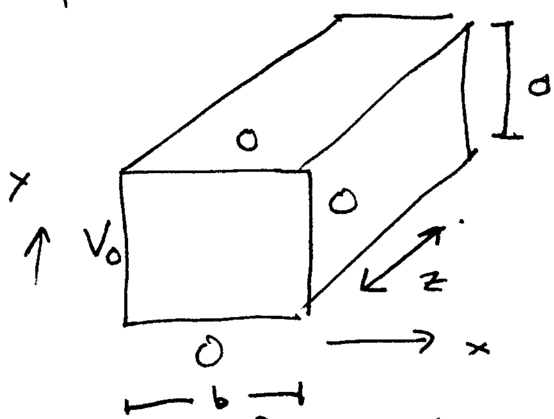
So $$E_x = -\frac{\partial V}{\partial x} = \frac{4V_0}{\pi} \sum_{n\ odd} \frac{1}{n(1-e^{-2k_n b})}$$

$$\times \left(-k_n e^{-k_n x} - k_n e^{k_n(x-2b)}\right) \sin k_n y$$

and $$\sigma_{left} = \varepsilon_0 \hat{x} \cdot \vec{E}(0) = -\varepsilon_0 \frac{\partial V}{\partial x}\bigg|_0$$

$$\sigma_{left} = \frac{4V_0}{\pi} \sum_{n\ odd} \frac{-k_n\left(1+e^{-2k_n b}\right)}{n\left(1-e^{-2k_n b}\right)} \sin k_y y$$

Ex Same problem in two-dimension



One face of an infinity long rectangular solid is held at V_0. The other faces are held at $V=0$.

Compute the field and charge.

$$V(x,y) = \sum\sum X_k Y_g$$

The y function must be zero at both $y=0$, and $y=a$.

We can meet this condition with the $\sin kx$ solution

if $\quad k = \frac{n\pi}{a}$ where n is an integer.

With this choice the x solutions become

$$e^{kx} \text{ and } e^{-kx}$$

The e^{-kx} solutions are more reasonable but we may need both to satisfy the boundary conditions

Our solution thus far is

$$V(x,y) = \sum_{n=1}^{\infty} \sin k_n y \left(C_n^+ e^{k_n x} + C_n^- e^{-k_n x} \right)$$

$$V(0,y) = V_0 = \sum_{n=1}^{\infty} \left(C_n^+ + C_n^- \right) \sin k_n y$$

$$V(b,y) = 0 = \sum_{n=0}^{\infty} \sin k_n y \left(C_n^+ e^{k_n b} + C_n^- e^{-k_n b} \right)$$

Now what?

Completeness - Certain classes of solutions to Laplace's eqn (and others) are complete. That is any other function that is a solution to the equation with the some boundary conditions can be written as a linear combination of the complete functions.

If $\{ f_n(x) \}$ are complete, the any function $f(x)$ can be written $f(x) = \sum_n a_n f_n(x)$

Orthogonality

$$\int_0^a \sin k_n x \, \sin k_m x \, dx = \begin{cases} 0 & , \text{if } m \neq n \\ \frac{a}{2} & , \text{if } m = n \end{cases}$$

$$\equiv \frac{a}{2} \delta_{nm}$$

Kronecker delta

$$\delta_{nm} = \begin{cases} 0 & , \text{if } m = n \\ 1 & , \text{if } m \neq n \end{cases}$$

Therefore if we want a_n and

$$f(x) = \sum_n a_n \sin k_n x$$

$$\int_0^a f(x) \sin k_m x \, dx = \sum_n a_n \int_0^a \sin k_n x \, \sin k_m x \, dx$$

$$= \sum_n a_n \frac{a}{2} \delta_{nm}$$

$$= \frac{a}{2} a_m$$

$$a_m = \frac{2}{a}\int_0^a f(x)\sin k_m x\,dx$$

Let's apply this to our problem

$$V(0,y) = V_0 = \sum_{n=1}^{\infty}\left(C_n^+ + C_n^-\right)\sin k_n y$$

$$C_n^+ + C_n^- = \frac{2}{a}V_0\int_0^a \sin k_n y\,dy$$

$$C_n^+ e^{k_n b} + C_n^- e^{-k_n b} = \frac{2}{a}\int 0\,\sin k_n y\,dy = 0$$

$$\cancel{C_n^- = -e^{2k_n a}C_n^+}$$

$$\cancel{\left(1 - e^{2k_n a}\right)C_n^+ = \frac{2}{a}V_0\int_0^a \sin k_n y\,dy}$$

$$C_n^+ = -C_n^- e^{-2k_n b}$$

$$C_n^-\left(1 - e^{-2k_n b}\right) = \frac{2}{a}V_0\int_0^a \sin k_n y\,dy$$

$$k_n = \frac{n\pi}{a} \qquad u = k_n y$$

$$du = k_n\, dy$$

$$C_n^- = \frac{2V_0}{a(1-e^{-2k_n b})k_n} \int_0^{n\pi} \sin u \, du$$

$$= \frac{2V_0}{a(1-e^{-2k_n b})k_n} \Big[-\cos u \Big]_0^{n\pi}$$

$$= \frac{2V_0}{a(1-e^{-2k_n b})k_n} \begin{cases} 0 & n \text{ even} \\ 2 & n \text{ odd} \end{cases}$$

$$= \begin{cases} 0 & n \text{ even} \\ \dfrac{4V_0}{n\pi(1-e^{-2k_n b})} & n \text{ odd} \end{cases}$$

So we build up the constant potential along the an edge out of an infinite linear combination of sines. Naturally, this is simply a Fourier series.