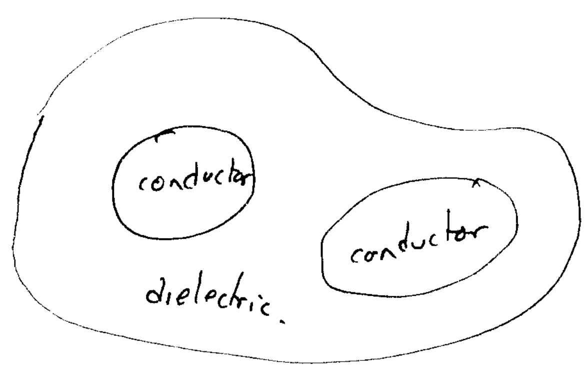
Energy and Capacitance

With our new field, $\vec{D}$, how does the potential difference get modified. The electric field is still the force per unit charge, so the work (per charge) to move a charge through an electric field is unchanged

$$\frac{W}{q} = -\int_{\vec{A}\to\vec{B}} \vec{E}\cdot d\vec{\ell} = V(\vec{B}) - V(\vec{A})$$

The insertion of a dielectric will modify the potential, but the potential is still calculated from the electric field.

Consider the following system



If charge $+Q$ is introduced on one conductor and charge $-Q$ on the other conductor, a field is produced and a potential difference will

The potential difference measured is proportional to the charge (because the field is proportional).

Capacitance (C)

$$C = \frac{Q}{|\Delta V|}$$

- Units Farads $1\,F = 1\,\frac{C}{V}$
- $\varepsilon_0 = 8.85 \times 10^{-12}\ F/m$
- Depends only on geometry and constants
  $\Rightarrow$ A configuration of conductors has an intrinsic property, capacitance, independent of any fields, charges, etc.

Energy (U) - The energy of a system is the work to build it charge by charge. Let's compute the energy of our conducting blob.

I. Begin with no charge.

II. The work to move a charge dQ is $dW = dQ\,\Delta V$ where ΔV is the potential difference between the conductors.

The total work and therefore the total energy is

$$U = W = \int_0^Q \Delta V \, dQ = C \int Q \, dQ$$

$$= \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} Q \Delta V = \frac{1}{2} C (\Delta V)^2$$

- This formula applies to any system that can be viewed as a capacitor, that is one or two equal and opposite surface charges.

- Note, this follows directly from our ~~form~~ potential form of the energy density

$$\frac{1}{2} \rho V$$

since all the charge is at the same potential

$$\frac{1}{2} Q V$$

Ex Let's use this to calculate stuff we already know how to do. Consider an isolated spherical conductor with radius R. If it is charged to a charge Q calculate the energy.

Method I

Since all the charge is at the same potential in reference to ∞, the energy is

$$U = \frac{1}{2}\int \rho V \, dv' = \frac{1}{2} Q V$$

$$V = \frac{kQ}{R}$$

$$U = \frac{1}{2} Q \left(\frac{kQ}{R} \right) = \frac{1}{2} k \frac{Q^2}{R}$$

Method II Integrate energy density

$$u = \frac{1}{2} \varepsilon_0 E^2$$

$$k = \frac{1}{4\pi\varepsilon_0}$$

$$\varepsilon_0 = \frac{1}{4\pi k}$$

$$= \frac{E^2}{8\pi k}$$

$$E = \frac{kQ}{r^2}$$

$$u = \left(\frac{1}{8\pi k} \right) \left(\frac{kQ}{r^2} \right)^2 = \frac{k}{8\pi} \frac{Q^2}{r^4}$$

Total Energy

$$U = \int u \, dv' = \int_R^{\infty} (4\pi r^2 dr) \frac{k}{8\pi} \frac{Q^2}{r^4}$$

using shell method with spherical shell $4\pi r^2 dr$

$$U = \frac{kQ^2}{2} \int_R^{\infty} \frac{dr}{r^2} = -\frac{kQ^2}{2r}\Bigg|_R^{\infty} = \frac{kQ^2}{2R}$$

Method III Use capacitance

Calculate capacitance of isolated sphere

I. Place Q on sphere

II $\vec{E} = \frac{kQ}{r^2}\hat{r}$

III $V = -\int E\,dr = \frac{kQ}{r} + C$

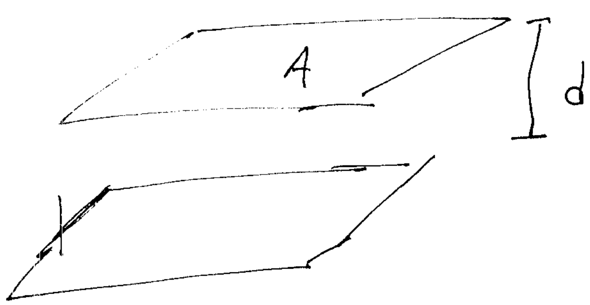
$$\Delta V = \frac{kQ}{R}$$

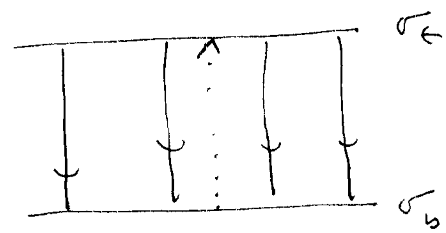
$$C = \frac{Q}{\Delta V} = \frac{R}{k} = 4\pi\varepsilon_0 R$$

Total Energy

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0 R} = \frac{1}{2}\frac{kQ^2}{R}$$

Parallel - Plate Capacitors





I Place $+Q$ on upper plate, $-Q$ on lower plate.

II Charge densities, $\sigma_t = \frac{Q}{A} \equiv \sigma \quad \sigma_b = \frac{-Q}{A}$

III Fields

$$\vec{E} = -\frac{\sigma}{\varepsilon_0}\hat{z}$$

IV $\Delta V = -\int \vec{E} \cdot d\vec{l} = Ed = \frac{\sigma}{\varepsilon_0} d$

V $C = \frac{Q}{\Delta V} = \frac{\sigma A}{\Delta V} = \frac{\sigma A}{\left(\frac{\sigma}{\varepsilon_0} d\right)}$

$$\boxed{C = \frac{A\varepsilon_0}{d}}$$

Alternate derivation of energy density. Consider a parallel-plate capacitor with separation d and area A.

The energy contained in the capacitor is

$$U = \frac{1}{2} C \Delta V^2 = \frac{1}{2}\left(\frac{A\varepsilon_0}{d}\right)(\Delta V)^2$$

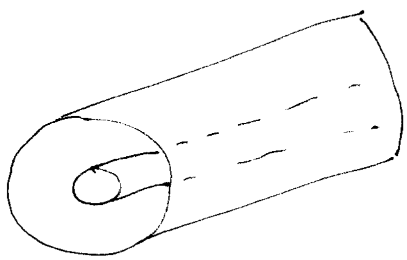
The energy per unit volume is then

$$u = \frac{U}{V} = \frac{U}{Ad} = \frac{1}{2}\varepsilon_0\left(\frac{\Delta V}{d}\right)^2$$

but $E = \frac{\Delta V}{d}$

$$u = \frac{1}{2}\varepsilon_0 E^2 \checkmark$$

Cylindrical Capacitors

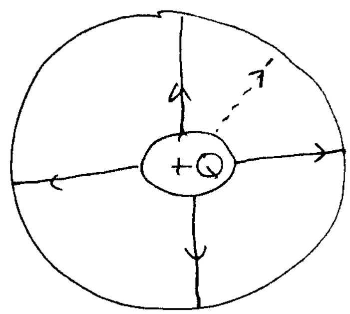


Inner radius r_i

Outer radius r_o

Consider a capacitor of length ℓ.

I. Place $+Q$ on inner conductor, $-Q$ on outer.



II Fields

Gauss $\quad \Phi = 2\pi\rho\ell E = \dfrac{Q_{enc}}{\varepsilon_0} = \dfrac{Q}{\varepsilon_0}$

$$E = \frac{1}{2\pi\varepsilon_0\rho}\left(\frac{Q}{\ell}\right) \qquad \text{Define } \lambda = \frac{Q}{\ell}$$

$$E = \frac{\lambda}{2\pi\varepsilon_0\rho}$$

$$\Delta V = -\int E\, d\rho = \frac{-\lambda}{2\pi\varepsilon_0}\int_{r_i}^{r_o} d\rho/\rho$$

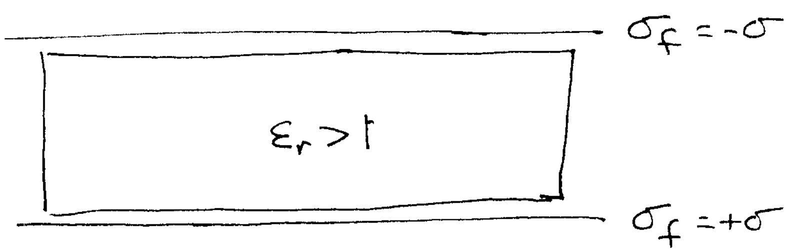
$$= \frac{-\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r_o}{r_i}\right)$$

$$|\Delta V| = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r_o}{r_i}\right)$$

$$C = \frac{Q}{|\Delta V|} = \frac{\lambda l}{|\Delta V|} = \frac{\lambda l}{\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r_i}{r_o}\right)}$$

$$\frac{C}{l} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{r_i}{r_o}\right)}$$

Now, add a dielectric to a capacitor. Let's start with a parallel plate capacitor.



I. Add $\pm Q$ $\Longrightarrow$ $\sigma_f = \sigma = Q/A$

II. $\vec{D} = \sigma \hat{z}$ as before.

III. $\Delta V = -\int \vec{E} \cdot d\vec{l}$ $\qquad\qquad \vec{E} = \dfrac{\vec{D}}{\varepsilon} = \dfrac{\sigma}{\varepsilon}\hat{z}$

$$= -\frac{\sigma}{\varepsilon_0}\int_0^d \cancel{\frac{\varepsilon}{\varepsilon_0}}\, dl = -\frac{\sigma d}{\varepsilon}$$

IV $\quad C = \dfrac{Q}{|\Delta V|} = \dfrac{\varepsilon A}{d} = \dfrac{\varepsilon_r \varepsilon_0 A}{d} = \varepsilon_r C_0$

$\Longrightarrow$ Dielectric increases capacitance by ε_r

Energy Density Parallel Plate

$$u = \frac{U}{V} = \frac{U}{Ad} = \frac{\frac{1}{2} C (\Delta V)^2}{dA}$$

$$= \frac{\frac{1}{2}\left(\frac{\varepsilon A}{d}\right)(\Delta V)^2}{dA} = \frac{1}{2}\varepsilon_0\left(\frac{\Delta V}{d}\right)^2$$

$$= \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\vec{E}\cdot\vec{D}$$