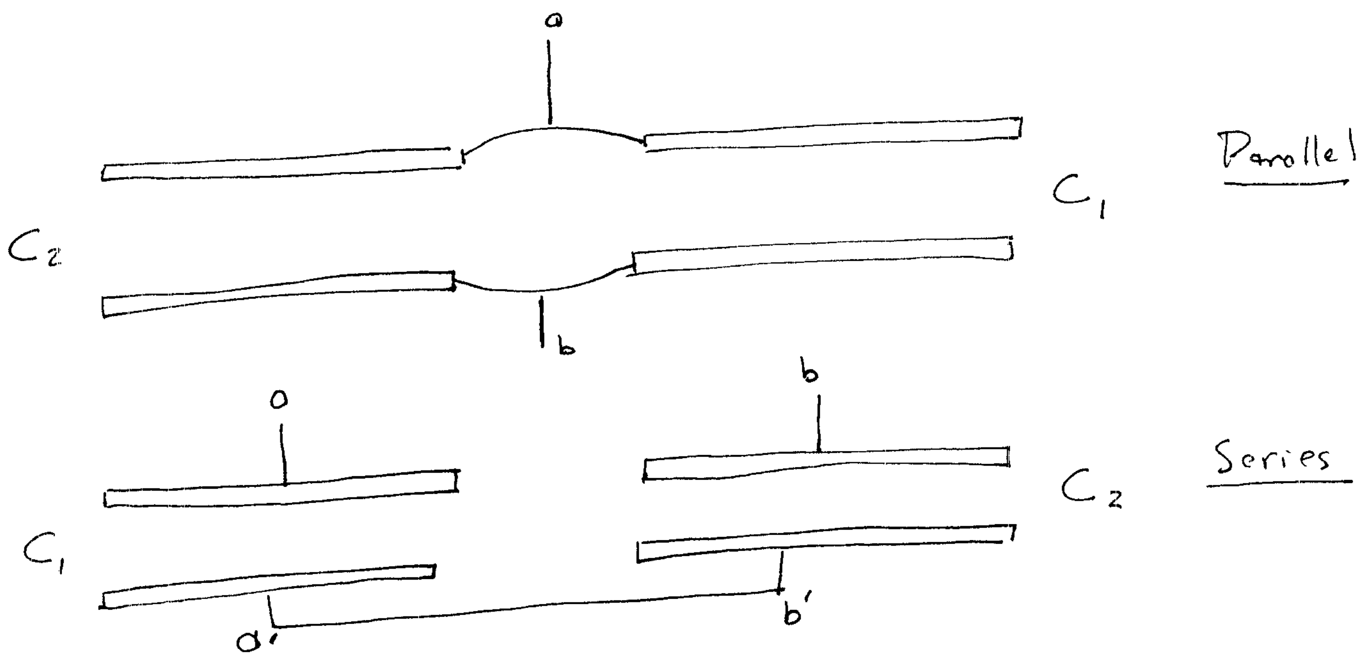
Capacitance II

Adding capacitors Once again consider our parallel-plate capacitor. We can connect two parallel-plate systems together in two different ways.



Let each capacitor have plate area A and separation d. The capacitance of the individual capacitors are $C_i = \frac{\varepsilon_0 A}{d}$

If we place $Q_a = +Q$ and $Q_b = -Q$ on the parallel combination, the charge divides between the two capacitors. The charge divides in such a way as to equalize the potential differences.

Let ΔV_{eq} be the final potential difference. This must be the same potential difference as the individual capacitors.

$$\Delta V_{eq} = \Delta V_1 = \Delta V_2$$

Using definition of capacitance,

$$Q_1 = C_1 \Delta V_1 \qquad Q_2 = C_2 \Delta V_2$$

$$Q = Q_1 + Q_2 = C_1 \Delta V_1 + C_2 \Delta V_2$$

$$= (C_1 + C_2) \Delta V_{eq}$$

$$C_{eq} = \frac{Q}{\Delta V_{eq}} = C_1 + C_2$$ Parallel

<u>Series</u> For the series combination, if $+Q$ is placed on plate A, $-Q$ is drawn to plate a' leaving $+Q$ on plate b' which balances a charge $-Q$ placed on b.

Now, the total charge on each capacitor is the same

$$Q_{eq} = Q_1 = Q_2$$

The total potential difference is the sum of the potential differences

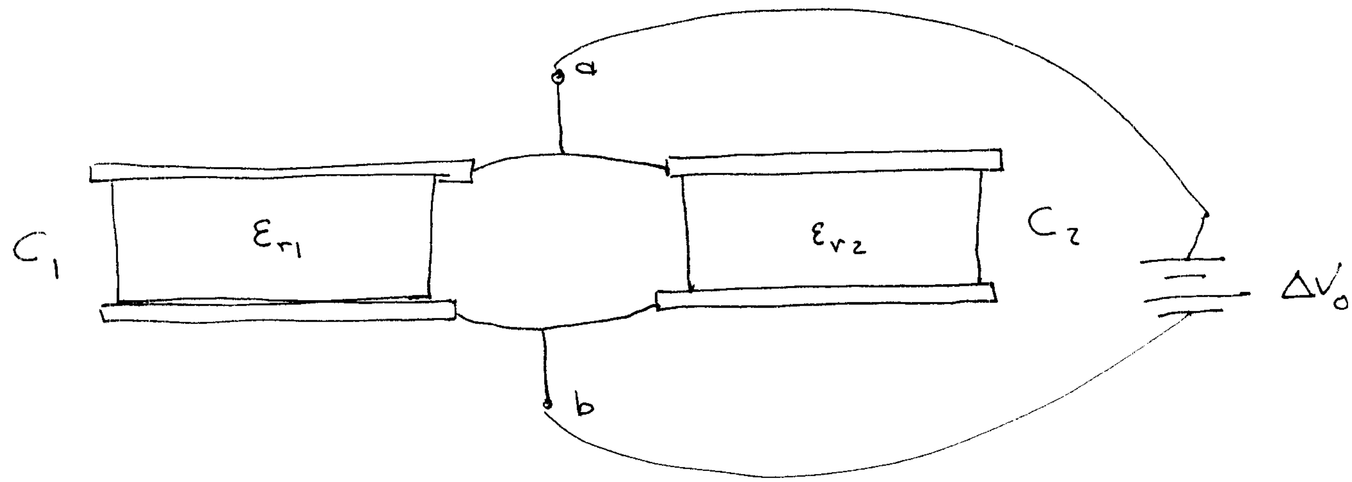
$$\Delta V = \Delta V_1 + \Delta V_2$$

$$= \frac{Q_1}{C_1} + \frac{Q_2}{C_2} = Q_{eq}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

$$C_{eq} = \frac{Q_{eq}}{\Delta V} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

$$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}}$$ Series capacitors.

So let's start adding some more interesting combinations



Let $C_0 = \dfrac{A\varepsilon_0}{d}$ be air-filled capacitance.

$$C_2 = \varepsilon_{r2} C_0$$

$$C_1 = \varepsilon_{r1} C_0$$

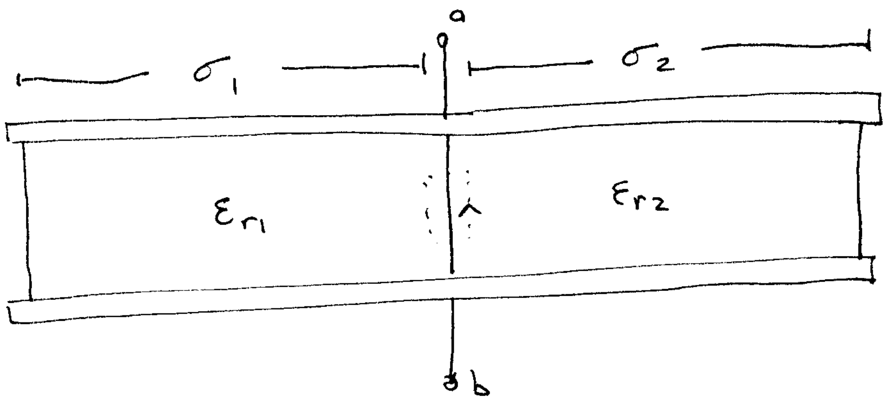
$$\Delta V_1 = \Delta V_2 = \Delta V_0$$

$$E_1 = E_2 = \frac{\Delta V_0}{d}$$

$$\sigma_1 = \frac{Q_1}{A} = \frac{C_1 \Delta V_1}{A} = \varepsilon_{r1} \frac{C_0 \Delta V_0}{A}$$

$$\sigma_2 = \varepsilon_{r2} \frac{C_0 \Delta V_0}{A}$$

Slide the two plates together



We must have the same system. Check boundary conditions.

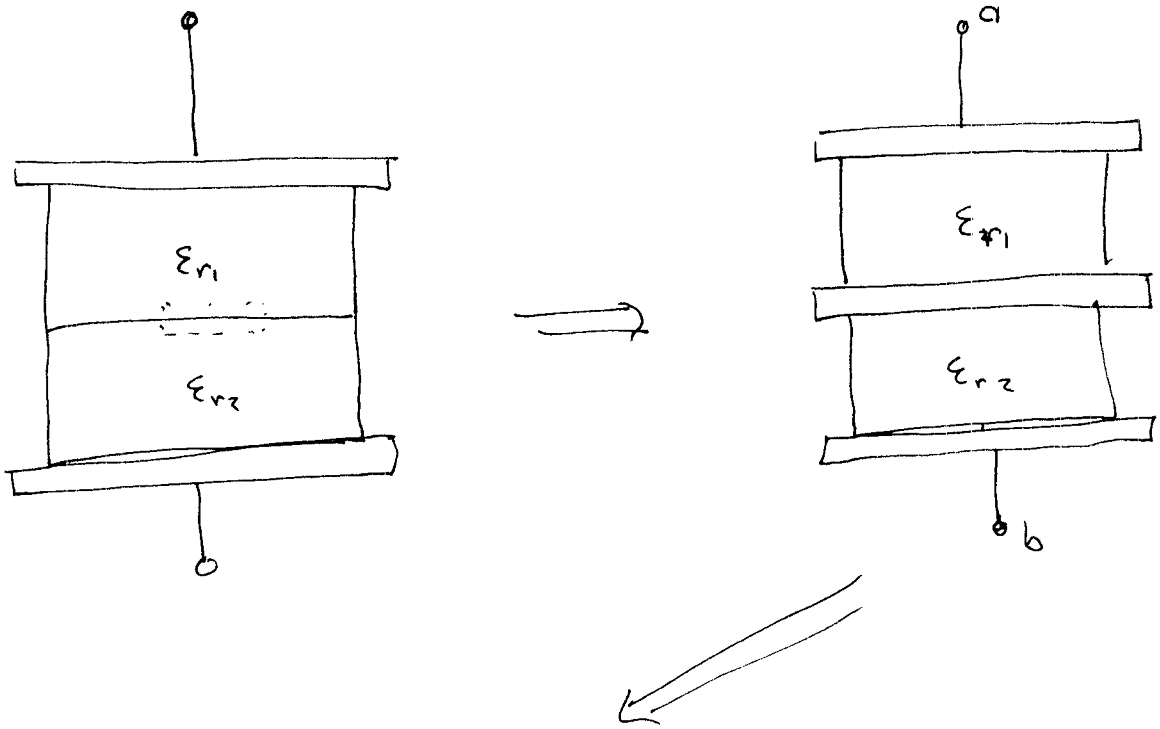
$$\vec{E}_1 \cdot \hat{n} = 0 = \vec{E}_2 \cdot \hat{n}$$

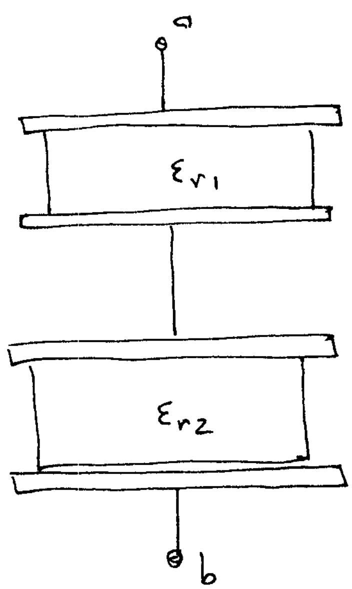
$$\vec{E}_1 \cdot \hat{t} = \vec{E}_2 \cdot \hat{t} \quad \text{since} \quad \vec{E}_1 = \vec{E}_2$$

$$C_{eq} = C_1 + C_2$$

Note, different charge densities above the two dielectrics.

Likewise

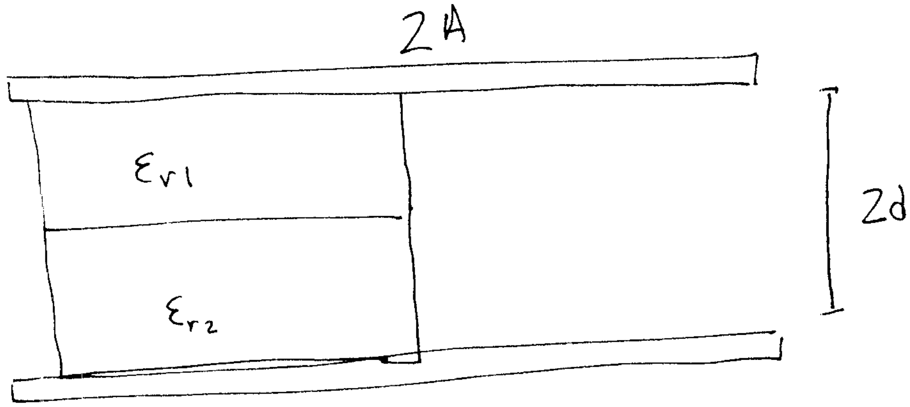




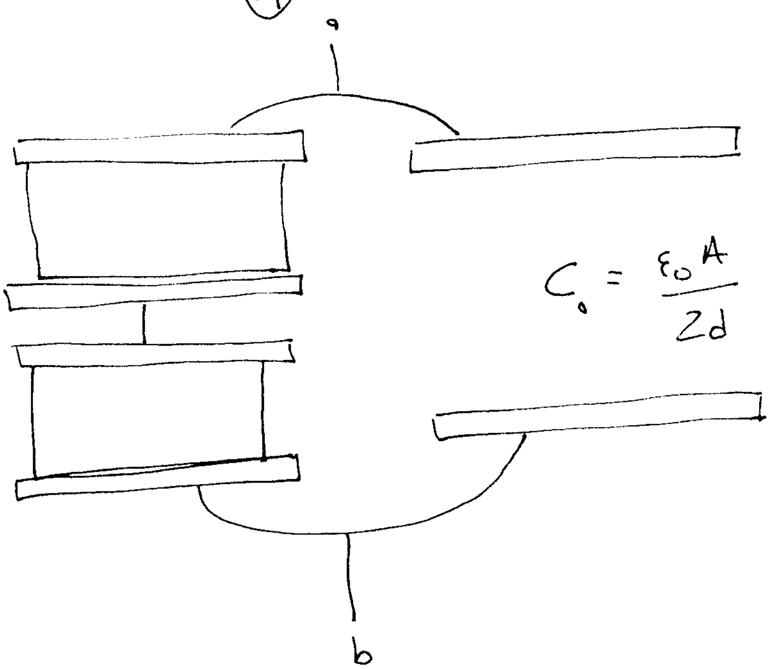
$$\frac{1}{C_{ab}} = \frac{1}{C_1} + \frac{1}{C_2}$$

Now, $\Delta V_{ab} = \Delta V_1 + \Delta V_2$

Next trick



$\Downarrow$



$$C_0 = \frac{\varepsilon_0 A}{2d} \qquad E_0 = \frac{\Delta V}{2d}$$

$$\frac{1}{C_{12}} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$Q_{12} = C_{12}\,\Delta V$$

$$\Delta V_1 = \frac{Q_{12}}{C_1} \qquad \Delta V_2 = \frac{Q_{12}}{C_2}$$

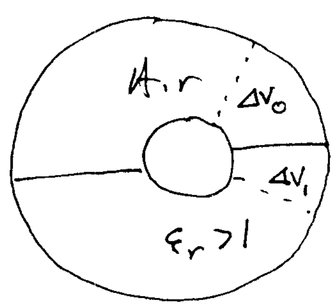
$$E_1 = \frac{\Delta V_1}{d} \qquad E_2 = \frac{\Delta V_2}{d}$$

Note, we no longer meet electrostatic boundary conditions

$$E_1 \neq E_2 \neq E_0$$

so we have to make the approximation that the fringing field is small.

$\Rightarrow$ If we do meet the electrostatic boundary conditions the solution is exact. For example,



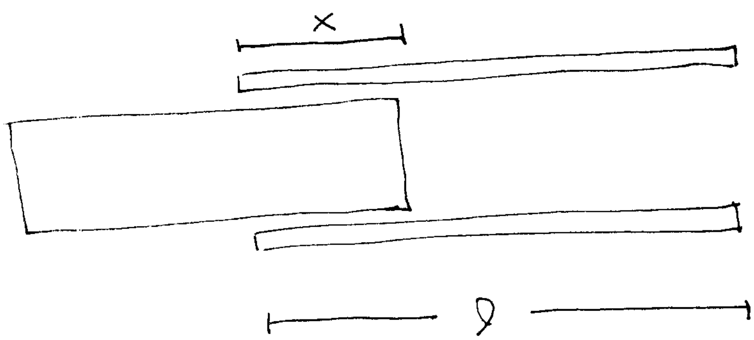
cylinder

$\Delta V_0 = \Delta V_1 \Rightarrow E_1 = E_0$ so boundary conditions are met.

$\Rightarrow$ Different charge densities

Force on Capacitors

Consider a dielectric partially inserted into a parallel-plate capacitor.



Let other plate dimension also be l

Two cases

I. Connected to battery (ΔV = constant)

II. Charged then disconnected (Q = constant)

Case II $\quad Q$ = constant

$$F = -\frac{dW}{dx} = -\frac{d}{dx}\frac{Q^2}{2C} = -Q^2\frac{d\,C^{-1}}{dx}$$

$$= \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx}$$

The two halves of the capacitor are in parallel

$$C = C_1 + C_2$$

$$= \varepsilon_r \frac{\varepsilon_0 \ell x}{d} + \frac{\varepsilon_0 \ell}{d}(\ell - x) = C_0\left(\varepsilon_r \frac{x}{\ell} - \frac{x}{\ell} + 1\right)$$

$$= \frac{\varepsilon_0 \ell}{d}\left(\varepsilon_r x + \ell - x\right) = C_0\left((\varepsilon_r - 1)\frac{x}{\ell} + 1\right)$$

$$\frac{dC}{dx} = \frac{(\varepsilon_r - 1)\varepsilon_0 \ell}{d} = \frac{C_0}{\ell}(\varepsilon_r - 1)$$

$$F = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx} = \frac{Q^2}{2C_0^2\left((\varepsilon_r - 1)\frac{x}{\ell} + 1\right)} \cdot \frac{C_0}{\ell}(\varepsilon_r - 1)$$

$$= \frac{Q^2}{2C_0 \ell} \frac{\varepsilon_r - 1}{(\varepsilon_r - 1)\frac{x}{\ell} + 1}$$

Case I $\quad \Delta V$ constant

Two sources of energy - Energy stored in capacitor and energy delivers by battery.

I. Capacitor $\quad W = \frac{1}{2} C (\Delta V)^2$

$$\frac{dW}{dx} = \frac{(\Delta V)^2}{2} \frac{dC}{dx}$$

II Battery $\quad dW = -V\,dQ$ $\qquad$ (- because battery loses energy

$$= -V(V\,dC)$$

$$\frac{dW}{dx} = -V^2 \frac{dC}{dx}$$

As before,

$$C = C_0\left((\varepsilon_r - 1)\frac{x}{l} + 1\right)$$

$$F = -\frac{dW}{dx} = -\frac{1}{2} V^2 \frac{dC}{dx} + V^2 \frac{dC}{dx}$$

$$= \frac{V^2}{2} \frac{dC}{dx} = \frac{C_0 V^2}{2D} (\varepsilon_r - 1)$$