Conservation Laws (Energy)

We have already established that local conservation of charge is contained in Maxwell's eqns

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$$

What about conservation of energy, momentum, and angular momentum?

We have shown the energy density in the electric or magnetic fields is

$$\frac{1}{2} \varepsilon_0 E^2 \quad \text{or} \quad \frac{1}{2\mu_0} B^2$$

so we might guess (and cross our fingers) that the energy density in the electromagnetic field is

$$U_{em} = \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2$$

If this is true, we should be able to show it is conserved locally, that is changes in the energy in a volume are accompanied by a flow of energy out of the volume.

Consider a system of charges q_i that produce fields $\vec{E}$ and $\vec{B}$ at time t. These fields exert forces on the charges doing work and changing the mechanical energy (kinetic + potential) of the charges. The work done on the charges by the fields is

$$dW = \vec{F} \cdot d\vec{l}$$

where $d\vec{l}$ is the displacement of the charges during dt.

The velocity is defined as

$$\vec{v} = \frac{d\vec{l}}{dt}$$

$$d\vec{l} = \vec{v}\, dt$$

So the work is

$$dW = \vec{F} \cdot \vec{v}\, dt$$

Substitute in the Lorentz force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

$$dW = \vec{F} \cdot \vec{v}\, dt = q(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{v}\, dt$$

$$= q\vec{v} \cdot \vec{E}\, dt = \vec{J} \cdot \vec{E}\, dt$$

since $\vec{J} = q\vec{v}$

$$\frac{dW}{dt} = \vec{E} \cdot \vec{J}$$

The total work done on the charges in the volume V,

$$\frac{dW}{dt} = \int_V \vec{E} \cdot \vec{J}\, d\tau$$

where $d\tau$ is the volume element.

Now, eliminate $\vec{J}$ using Ampere's Law.

Ampere $\quad \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

$$\vec{J} = \frac{1}{\mu_0} \nabla \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Substitute

$$\frac{dW}{dt} = \int_V \vec{E}\cdot J \, d\tau = \frac{1}{\mu_0}\int_V \vec{E}\cdot(\nabla\times\vec{B})\,d\tau - \int_V \varepsilon_0 \vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\,d\tau$$

*

Work on each term

$$* \quad \frac{\partial\, \vec{E}\cdot\vec{E}}{\partial t} = \frac{1}{2}\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{\partial E^2}{\partial t}$$

$$\vec{E}\cdot(\nabla\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{E}) - \nabla\cdot(\vec{E}\times\vec{B}) \qquad \left[\begin{array}{c}\text{Vector}\\ \text{identity}\end{array}\right]$$

Use Faraday's Law

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{E}\cdot(\nabla\times\vec{B}) = -\vec{B}\cdot\frac{\partial\vec{B}}{\partial t} - \nabla\cdot(\vec{E}\times\vec{B})$$

$$= -\frac{\partial B^2}{\partial t} - \nabla\cdot(\vec{E}\cdot\vec{B})$$

Substitute everything back,

$$\frac{dW}{dt} = -\frac{d}{dt}\int_V \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) d\tau$$

$$-\frac{1}{\mu_0}\int_V \nabla \cdot \left(\vec{E} \times \vec{B}\right) d\tau$$

Use divergence thm,

$$\frac{dW}{dt} = -\frac{d}{dt}\int_V u_{em}\, d\tau - \oint_S \vec{S} \cdot d\vec{a}$$

surface

where the energy density is

$$u_{em} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

and I have introduced the Poynting Vector

$$\boxed{\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} \quad \text{Poynting Vector}}$$

From the form of the conservation expression, $\vec{S}$ is the flow of energy.

If we write dW/dt, the mechanical energy as a density u_{mech} so that

$$\frac{dW}{dt} = \int_V u_{mech}\, d\tau$$

We can write the conservation equation for energy

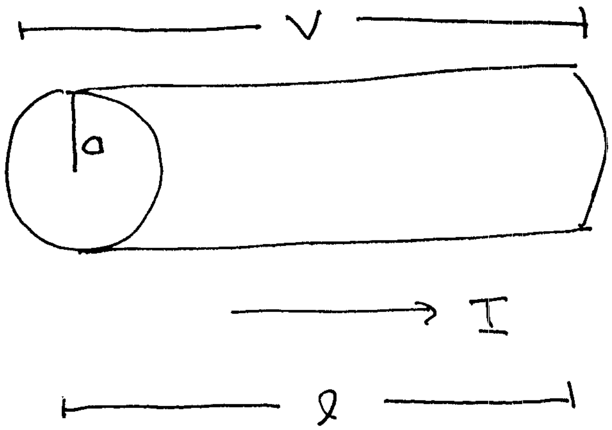
$$\frac{d}{dt}\int_V (u_{em} + u_{mech})\, d\tau = -\oint_S \vec{S}\cdot d\vec{a}$$

$$= -\int_V \nabla\cdot\vec{S}\, d\tau$$

– or –

$$\boxed{\frac{\partial}{\partial t}(u_{mech} + u_{em}) = -\nabla\cdot\vec{S}}$$

Example (Griffiths 8.1) Calculate the energy / time lost to heat in a wire that carries a current I driven by a potential V.



The potential is generated by a field $E = \frac{V}{l}$.

The current produces a magnetic field at the outer edge of the wire of

$$B = \frac{\mu_0 I}{2\pi a}$$

The two fields are at right angles so

$$|\vec{S}| = \frac{1}{\mu_0} |\vec{E} \times \vec{B}| = \frac{1}{\mu_0} \left(\frac{V}{l} \right) \left(\frac{\mu_0 I}{2\pi a} \right)$$

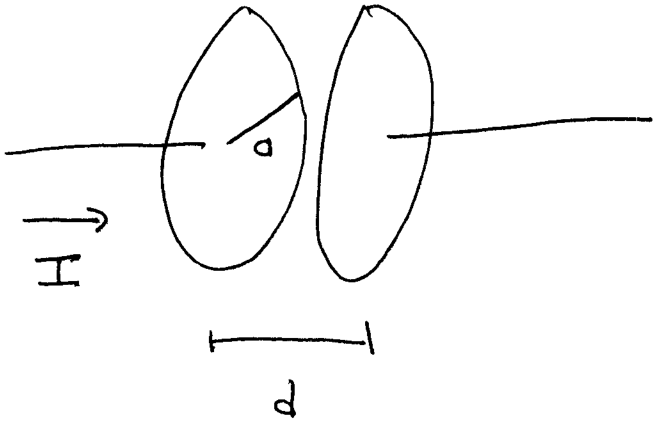
$$= \frac{VI}{2\pi a l}$$

The total energy flow out the sides of the wire is

$$\text{Power} = |\vec{S}|\ \text{Area} = |\vec{S}|\, 2\pi a \ell$$

$$= IV \quad \checkmark$$

Griffith's 8.2 Charging Capacitor



Interior to the capacitor, there is no physical current but the displacement current is equal to the applied current, $I = I_d$, and therefore the magnetic field is

$$B = \frac{\mu_0 I}{2\pi a}$$

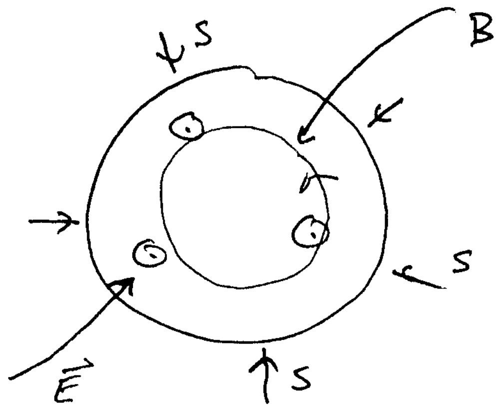
at the edge.

The electric field points from the left plate to the right plate and has magnitude

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}$$

$A \equiv$ Plate Area

End view



So energy is flowing into the sides of the capacitor.

$$|\vec{S}| = \frac{1}{\mu_0} |\vec{E} \times \vec{B}| = \frac{1}{\mu_0} \left(\frac{\mu_0 I}{2\pi a} \right) \left(\frac{Q}{\varepsilon_0 A} \right)$$

$$= \frac{I Q}{2\pi a A \varepsilon_0}$$

The energy stored in the capacitor is

$$U = \frac{1}{2} \frac{Q^2}{C}$$

and the power

$$\frac{dU}{dt} = \frac{Q}{C} \frac{dQ}{dt} = V I$$

The total power into the capacitor is

$$\text{Power} = |\vec{S}|\ \text{Area}$$

$$= |\vec{S}|\ 2\pi a\, d$$

$$= \frac{I\,Q\,d}{A\,\varepsilon_0} = I\,E\,d$$

$$= I\,V \quad \checkmark$$