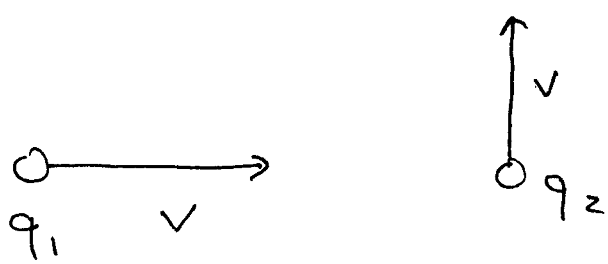
Conservation Laws (Momentum)

Newton's Third Law appears not to work in electromagnetism. It is fairly easy to find configurations of moving charge where $\vec{F}_{AB} \neq -\vec{F}_{BA}$. Given the right semester, I probably had you do it in UPII. Consider,



The field from 1 at 2 is zero, but the field and force of 2 at 1 is maximum.

Newton III is just a statement of the local conservation of momentum. The universe would not work for very long if momentum leaked out of the electromagnetic field.

The forces we considered above were the mechanical forces in analogy to dW/dt as the mechanical energy. We can save the universe if, like energy, some momentum is stored in the fields.

To work with momentum $\vec{P}$, we state with force since $\vec{F} = d\vec{P}/dt$.

The total force exerted by the fields in a volume V on the charges in the volume is

$$\vec{F} = \int_V \rho\left(\vec{E} + \vec{v}\times\vec{B}\right) d\tau$$

$$= \int_V \left(\rho\vec{E} + \vec{J}\times\vec{B}\right) d\tau$$

where ρ is the charge density an $\rho\vec{v} = \vec{J}$ is the current density.

The force density, force per unit volume, is

$$\vec{f} = \rho \vec{E} + \vec{J} \times \vec{B}$$

Now, as before, we use Maxwell's eqns to eliminate ρ and $\vec{J}$.

Gauss $\qquad \varepsilon_0 \nabla \cdot \vec{E} = \rho$

Ampere $\qquad \nabla \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \dfrac{\partial \vec{E}}{\partial t}$

$$\vec{f} = \varepsilon_0 (\nabla \cdot \vec{E}) \vec{E} + \frac{1}{\mu_0} \left(\nabla \times \vec{B} - \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \times \vec{B}$$

Let's work on it. We know the combination $\vec{E} \times \vec{B}$ is important.

$$\frac{\partial}{\partial t} \vec{E} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} \times \vec{B} + \vec{E} \times \frac{\partial \vec{B}}{\partial t}$$

Use Faraday's Law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right) = \frac{\partial \vec{E}}{\partial t}\times\vec{B} - \vec{E}\times\left(\nabla\times\vec{E}\right)$$

So

$$\frac{\partial E}{\partial t}\times\vec{B} = \frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right) + \vec{E}\times\left(\nabla\times\vec{E}\right)$$

$$\begin{aligned}\vec{f} = \varepsilon_0\left(\nabla\cdot\vec{E}\right)\vec{E} &- \frac{1}{\mu_0}\vec{B}\times\left(\nabla\times\vec{B}\right)\\ &- \varepsilon_0\vec{E}\times\left(\nabla\times\vec{E}\right)\\ &- \varepsilon_0\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right)\end{aligned}$$

Where I used $\left(\nabla\times\vec{B}\right)\times\vec{B} = -\vec{B}\times\left(\nabla\times\vec{B}\right)$

The expression is sort of symmetric and we can make it look better by adding $\left(\nabla\cdot\vec{B}\right)\vec{B} = 0$.

Keep working   Use a vector identity on

$$\nabla(E^2) = 2(\vec{E}\cdot\nabla)\vec{E} + 2\vec{E}\times(\nabla\times\vec{E})$$

$$\nabla(B^2) = 2(\vec{B}\cdot\nabla)\vec{B} + 2\vec{B}\times(\nabla\times\vec{B})$$

$$\vec{f} = \varepsilon_0\left((\nabla\cdot\vec{E})\vec{E} + (\vec{E}\cdot\nabla)\vec{E}\right)$$

$$+ \frac{1}{\mu_0}\left((\nabla\cdot\vec{B})\vec{B} + (\vec{B}\cdot\nabla)\vec{B}\right)$$

$$- \frac{1}{2}\nabla\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)$$

$$- \varepsilon_0 \frac{\partial}{\partial t}\vec{E}\times\vec{B}$$

The third and fourth line are recognizable as the gradient of the energy density and the time rate of change of the energy flow.

This mess can be repackaged using the

Maxwell Stress Tensor

$$T_{ij} = \varepsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right)$$

where δ_{ij} is the Kronecker delta

- A tensor is a matrix that transforms like a vector. So the stress tensor is basically a 3×3 matrix.
- We're looking for a force density $\vec{f}$, so we will have to multiply $\overleftrightarrow{T}$ by a vector to get something that looks like a vector. If $\vec{a}$ is a vector then

$$\left(\vec{a} \cdot \overleftrightarrow{T} \right)_j = \sum_i a_i T_{ij}$$

so $\vec{a} \cdot \overleftrightarrow{T}$ is a vector.

What are we hoping happens? We would like to break the force density (the time rate of change of momentum density) into a term representing the storage of momentum in the fields and a term representing the flow of momentum.

The mess we had earlier can be written in terms of the stress tensor

$$\vec{f} = \nabla \cdot \overleftrightarrow{T} - \varepsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t}$$

or the total force (using the divergence thm)

$$\vec{F} = \int_V \vec{f}\, d\tau = \oint_S \overleftrightarrow{T} \cdot d\vec{a} - \varepsilon_0 \mu_0 \frac{d}{dt} \int_V \vec{S}\, d\tau$$

Now let's recover conservation of momentum ($\vec{p}_{mech}$)

$$\vec{F} = \frac{d\vec{p}_{mech}}{dt} = -\varepsilon_0 \mu_0 \frac{d}{dt} \oint_V \vec{S}\, d\tau + \oint_S \overleftrightarrow{T} \cdot d\vec{a}$$

The first term is the momentum stored in the fields

$$\vec{p}_{em} = \mu_0 \varepsilon_0 \int_V \vec{S}\, d\tau$$

The second term is the momentum flux out of the surface, so $-\overleftrightarrow{T}$ is the momentum flux density.

From $\vec{p}_{em}$, the momentum density in the fields is

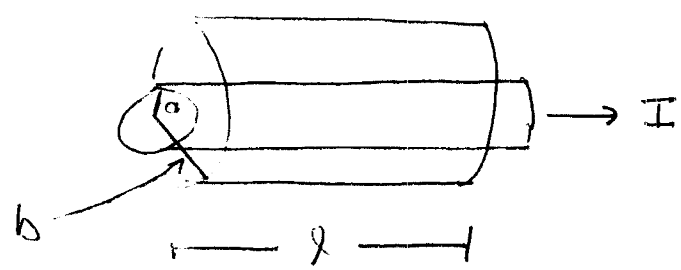
$$\vec{\wp}_{em} = \mu_0 \varepsilon_0 \vec{S}$$

Conservation of Momentum with E+M fields becomes

$$\frac{d}{dt}\left(\vec{\wp}_{mech} + \wp_{em}\right) = \nabla \cdot \overleftrightarrow{T}$$

- Note missing negative sign on $\nabla \cdot \overleftrightarrow{T}$
- $\wp_{mech}$ is the mechanical momentum density

Ex - Griffiths 8.3 Calculate the momentum stored in the fields in a co-axial cable carrying current I. The inner conductor carries charge density λ.



The electric field in the region between the conductors is

$$E = \frac{\lambda}{2\pi\epsilon_0 r}$$

and the magnetic field is

$$B = \frac{\mu_0 I}{2\pi r}$$

The electric and magnetic fields are at right angles so

$$S = \frac{1}{\mu_0} EB = \frac{1}{\mu_0}\left(\frac{\lambda}{2\pi\epsilon_0 r}\right)\left(\frac{\mu_0 I}{2\pi r}\right)$$

$$= \frac{\lambda I}{4\pi^2\epsilon_0 r^2}$$

This makes sense since a cable should have a flux of energy down the axis.

The power flowing down the cable is

$$P = \int \vec{S} \cdot d\vec{a} = \int_a^b 2\pi r \, S \, dr$$

$$= \frac{\lambda I}{2\pi \varepsilon_0} \int_a^b \frac{1}{r} dr$$

$$= \frac{\lambda I}{2\pi \varepsilon_0} \ln(b/a)$$

The potential difference between the conductors is

$$V = -\int_a^b E \, dr = \frac{\lambda}{2\pi \varepsilon_0} \ln(b/a)$$

so $P = IV$ ✓

The momentum stored in the fields is

$$\vec{p}_{em} = \mu_0 \varepsilon_0 \int \vec{S} \, d\tau = \mu_0 \varepsilon_0 \hat{z} \int_a^b 2\pi r l S \, dr$$

where $\hat{z}$ points down the cable.

$$\vec{P}_{em} = \frac{\mu_0 \lambda I}{4\pi^2} \hat{z} \int \frac{J\, 2\pi r\, dr}{r^2}$$

$$= \frac{\mu_0 \lambda I J}{2\pi} \ln(b/a)\, \hat{z}$$

Angular Momentum — The angular momentum density in the fields is

$$\vec{J}_{em} = \vec{r} \times \mathcal{P}_{em} = \varepsilon_0 \left(\vec{r} \times (\vec{E} \times \vec{B}) \right)$$

and naturally angular momentum is also conserved.