Cylindrical Coordinates (z = constant)

$$\nabla^2 V = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

$$V = P(\rho)\,\Phi(\phi)$$

$$\frac{1}{P(\rho)}\,\rho\,\frac{\partial}{\partial \rho}\left(\rho \frac{\partial P}{\partial \rho}\right) = -\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = k^2$$

$$\frac{d^2 \Phi}{d\phi^2} + k^2 \Phi = 0$$

$$\Phi = \sin k\phi \,,\quad \cos k\phi$$

To be continuous, $\sin k 2\pi = \sin 0 \qquad \cos k 2\pi = \cos \theta$

$\Longrightarrow \quad k = n$ some integer.

$$\rho \frac{\partial}{\partial \rho}\left(\rho \frac{\partial P}{\partial \rho}\right) - n^2 P = 0$$

Try $\quad P = \rho^a$

$$\rho \frac{\partial}{\partial \rho} a \rho^a \quad - n^2 P = 0$$

$$a^2 \rho^a - n^2 \rho^a = 0$$

$n = a \qquad n = -a$

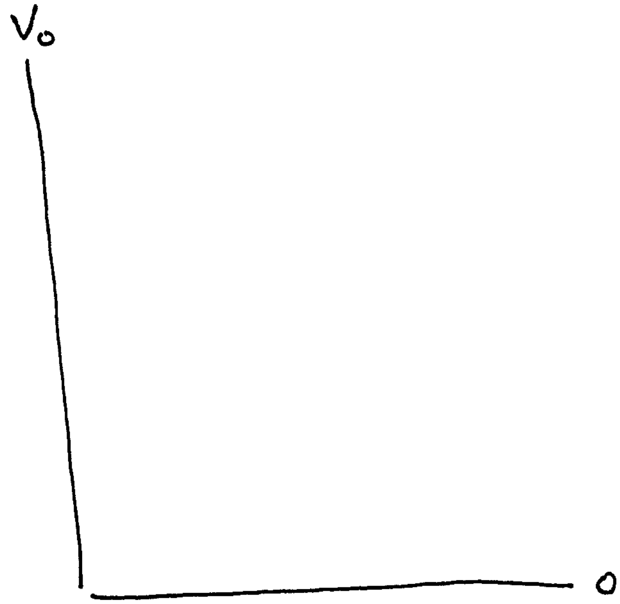
Solutions

$1 \qquad \ln \rho \qquad \phi$

$\rho^n \cos n\phi \qquad \rho^{-n} \cos n\phi$

$\rho^n \sin n\phi \qquad \rho^{-n} \sin n\phi$

Ex



Sln

$\cos n\phi$ is out because $V(\rho, 0) = 0$

ρ^{+n} are out because they blow up.

1, $\ln \rho$ don't meet boundary conditions.

$$V(\rho, \phi) = \sum_n a_n \rho^{-n} \sin n\phi$$

$$V(\rho, \pi/2) = \sum_n a_n \rho^{-n} \sin \frac{n\pi}{2} = V_0$$

Is not solvable.

Check the last trivial solution (which is not in the book).

$$V(\rho, \phi) = A\phi + B$$

$$V(\rho, 0) = 0 \Rightarrow B = 0$$

$$V(\rho, \pi/2) = V_0 \Rightarrow A = \frac{V_0}{\pi/2} = \frac{2V_0}{\pi}$$

$$V(\rho, \phi) = \frac{2V_0}{\pi}\phi$$

$$\vec{E} = -\nabla V = -\frac{2V_0}{\pi}\frac{1}{\rho}\frac{\partial\phi}{\partial\phi}\hat{\phi}$$

$$= -\frac{2V_0}{\pi\rho}\hat{\phi}$$

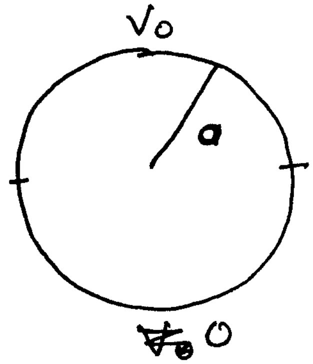
Charge Density on Plane - $\phi = 0$ plane

$$\sigma_0 = (\hat{n} \cdot \vec{E})\varepsilon_0 = \frac{-2V_0\varepsilon_0}{\pi\rho}$$

$$\sigma_{\pi/2} = \frac{2V_0}{\pi\rho}\varepsilon_0$$

Let's try something harder

Ex Cylinder with top half at V_0, bottom half at $V=0$. Find potential.



Solution is different if you are inside or outside.

Inside 1, $\ln\rho$, ϕ solutions bad.

$\rho^{-n}\cos n\phi$ and $\rho^{-n}\sin n\phi$ are bad since they blow up at zero.

$$V(\rho,\phi) = \sum_n a_n \rho^n \cos n\phi + b_n \rho^n \sin n\phi$$

Boundary Conditions

If $\cancel{-\pi/2}\ 0 < \phi < \pi\cancel{/2}$, $V(a,\phi) = V_0$,

$\pi < \phi < 2\pi$, $V(a,\phi) = 0$

$$V(a,\phi) = \begin{cases} V_0 & 0 \leq \phi \leq \pi \\ 0 & \pi \leq \phi < 2\pi \end{cases}$$

$$= \sum_n a_n a^n \cos n\phi + b_n a^n \sin n\phi$$

$$\int_0^{2\pi} \cos n\phi \cos m\phi \, d\phi = \delta_{nm} \pi \quad (\text{orthogonality})$$

Note $= 2\pi$ if $m = n = 0$

$$\int_0^{2\pi} \cos m\phi \, V(a,\phi) \, d\phi = a_m \pi a^m$$

$$\int_0^{2\pi} \sin m\phi \, V(a,\phi) \, d\phi = b_m \pi a^m$$

$$a_m = \frac{V_0}{\pi a^m} \int_0^{\pi} \cos m\phi \, d\phi = \frac{V_0}{\pi m a^m} \sin m\phi \Big|_0^{\pi} =$$

$$b_m = \frac{V_0}{\pi a^m} \int_0^{\pi} \sin m\phi \, d\phi = -\frac{V_0}{\pi m a^m} \cos m\phi \Big|_0^{\pi}$$

$$= \begin{cases} \dfrac{2V_0}{\pi m a^m} & m \text{ odd} \\ 0 & m \text{ even} \end{cases}$$

$$V(\rho, \phi) = \frac{2V_0}{\pi} \sum_{n\ odd} \frac{\rho^n}{n a^n} \sin n\phi$$