Lecture 4

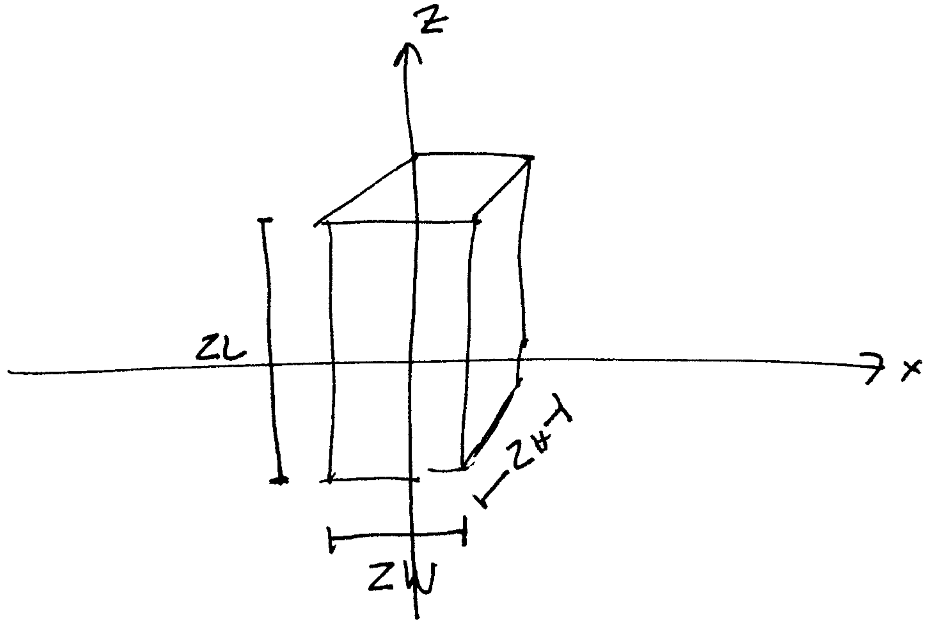
Revisiting UPII

Gauss' Law $\quad \oint_S \vec{E} \cdot d\vec{a} = \oint_S \vec{E} \cdot \hat{n}\, da = \frac{Q_{enc}}{\epsilon_0}$

Coulomb's $\quad \vec{E} = \frac{kq}{r^2} \hat{r}$

With what you know from UPII and the skills you mastered in Cal II and Cal III, you can do a lot of physics.

Consider



Compute field along axis

1 Decompose the problem into sub-problem -

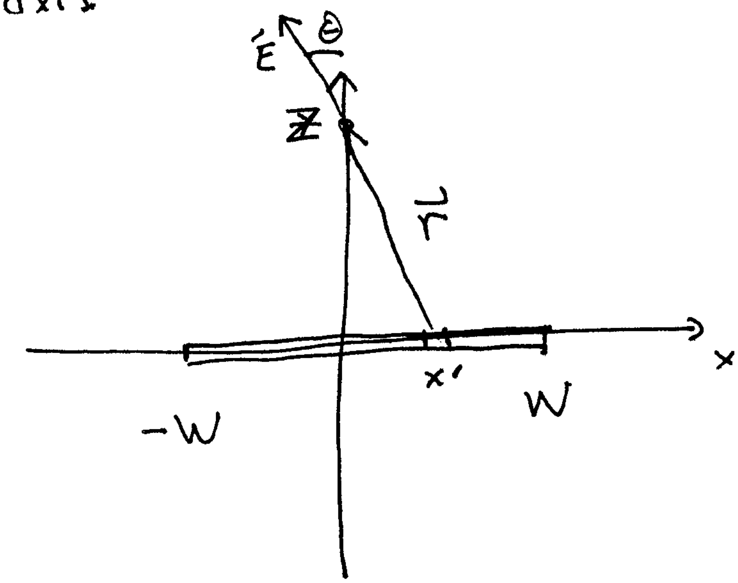
A Field of finite line

B Field of finite plane

C Field of volume

Check each step with limits.

Field of finite line charge occupying $-W$ to W on x-axis with linear charge density λ. Compute field on axis.



(1) By the reflection symmetry through y-z plane, the field must point along the axis.

(2) $\vec{E}(z) = \hat{z} \sum |E_i| \cos\theta$

$$|E_i| = \frac{k\,dq}{(x^2+z^2)} \qquad \cos\theta = \frac{z}{\sqrt{x^2+z^2}}$$

$$dq = \lambda\, dx$$

$$\vec{E}(z) = \hat{z} \int_{-W}^{W} \frac{z k \lambda\, dx}{(x^2+z^2)^{3/2}}$$

$$= z\hat{z} k\lambda \int_{-W}^{W} \frac{dx}{(x^2+z^2)^{3/2}}$$

$$= z\hat{z} k\lambda \left(\frac{2W}{z^2\sqrt{W^2+z^2}} \right) = \frac{2k\lambda\hat{z}}{z} \left(\frac{W}{\sqrt{W^2+z^2}} \right)$$

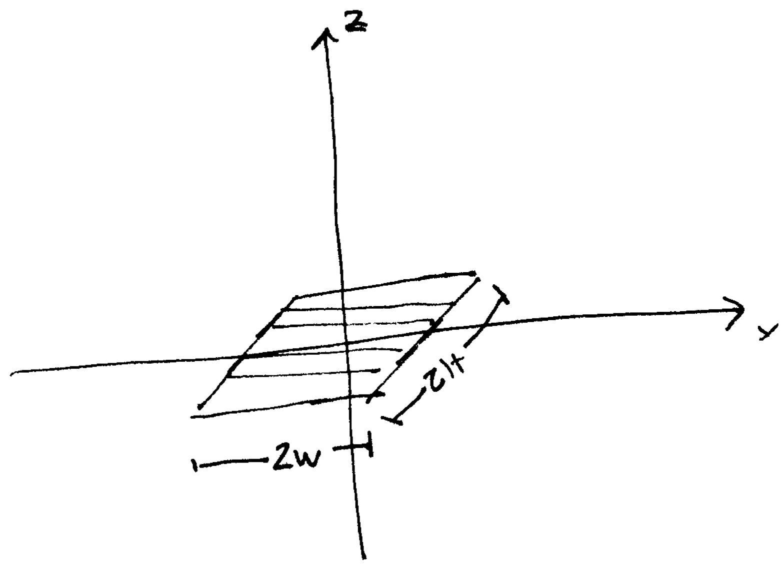
(3) Check $\lim_{W \to \infty} \frac{W}{\sqrt{W^2+z^2}} \to 1$

$$\vec{E} \to \frac{2k}{z}\hat{z} = \frac{2k\lambda}{z}\hat{z}$$

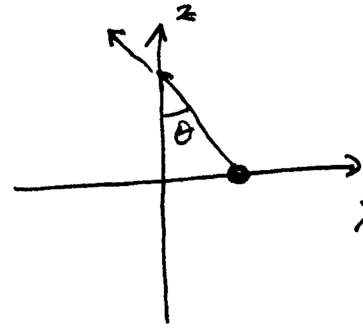
$$= \frac{\lambda}{2\pi\varepsilon_0 z}\hat{z}$$

Which is the field of an infinite line of charge.

Compute the field of a rectangle of charge with uniform surface charge density σ that occupies $-W \leq x \leq W$ and $-H \leq y \leq H$ in the x-y plane.



<u>Cut plate into strips</u>



<u>Field is along z-axis</u>

$$\vec{E} = \hat{z} \sum |E_{strip}(d)| \cos\theta$$

where $d = \sqrt{y^2+z^2}$ is the distance from the strip to the field point.

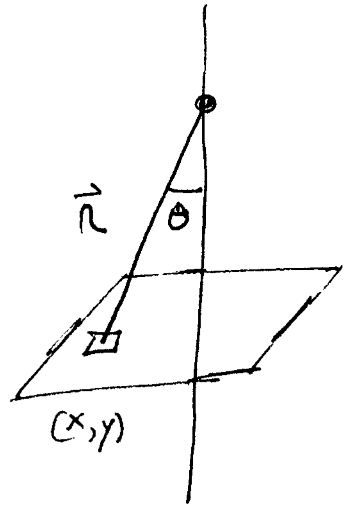
$$E_{strip}(d) = \frac{2k\lambda}{\sqrt{z^2+y^2}} \left(\frac{W}{\sqrt{W^2+z^2+y^2}} \right)$$

$$\cos\theta = \frac{z}{d} = \frac{z}{\sqrt{z^2+y^2}} \qquad \lambda = \sigma dy$$

<u>So</u> $$\vec{E} = \hat{z} \int_{-H}^{H} \frac{2k\lambda W \sigma dy}{\left(\sqrt{z^2+y^2}\right)\left(\sqrt{W^2+z^2+y^2}\right.}$$

This is a huge mess, let's try Calc III.

Other Method - 2d integral



$$\vec{r} = (0, 0, z) - (x, y, 0)$$

$$= (-x, -y, z)$$

$$\hat{r} = \sqrt{x^2+y^2+z^2}$$

$$\cos\theta = \hat{r} \cdot \hat{z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$dq = \sigma\, dx\, dy$

$$\vec{E} = \hat{z} \sum \frac{k\, dq}{\left(\sqrt{x^2+y^2+z^2}\right)^2} \cdot \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$= k z \hat{z} \sigma \int_{-W}^{W} dx \int_{-H}^{H} dy \frac{1}{(x^2+y^2+z^2)^{3/2}}$$

Result Let $W = H$,

$$\vec{E}(z) = \frac{2\hat{z}\,k\,z\,\sigma}{z}\left(-i\,\operatorname{arctanh}\left(\frac{W^2 + iWz + z^2}{W\sqrt{2W^2+z^2}}\right) + i\,\operatorname{arctanh}\left(\frac{W - iWz + z^2}{W\sqrt{2W^2+z^2}}\right)\right)$$

~~Limit $W \to \infty$ $(\;)$ = π~~

$$\cancel{\vec{E}(z) = 4\pi\sigma\hat{z} = \frac{\sigma}{2\varepsilon_0}\hat{z}} \quad \checkmark$$

Limit $W \to \infty$ $(\;) \to \pi$

$$\vec{E}(z) = 2\pi\hat{z}\,k\,\sigma = \frac{\sigma}{2\varepsilon_0}\hat{z} \quad \checkmark$$

Maple worksheet for integrals follows.

8

> $integrate\left(\frac{1}{(x^2+z^2)^{\left(\frac{3}{2}\right)}},\ x=-W..W\right)$ *assuming* $z>0$;

$$\frac{2\,W}{\sqrt{W^2+z^2}\ z^2} \tag{1}$$

> $integrate\left(\frac{1}{(x^2+z^2)^{\left(\frac{3}{2}\right)}},\ x=-W..W\right)$ *assuming* $z<0$;

$$\frac{2\,W}{\sqrt{W^2+z^2}\ z^2} \tag{2}$$

> $with(RealDomain)$;

```
Warning, these protected names have been redefined and unprotected:
Im, Re, `^`, arccos, arccosh, arccot, arccoth, arccsc, arccsch,
arcsec, arcsech, arcsin, arcsinh, arctan, arctanh, cos, cosh, cot,
coth, csc, csch, eval, exp, expand, limit, ln, log, sec, sech,
signum, simplify, sin, sinh, solve, sqrt, surd, tan, tanh
```

[$\Im$, $\Re$, ^, arccos, arccosh, arccot, arccoth, arccsc, arccsch, arcsec, arcsech, arcsin, arcsinh, arctan, arctanh, cos, cosh, cot, coth, csc, csch, *eval*, exp, *expand*, *limit*, ln, log, sec, sech, signum, *simplify*, sin, sinh, *solve*, sqrt, surd, tan, tanh] (3)

Perform the y integration

> $int\left(int\left(\frac{1}{(z^2+y^2)\cdot(y^2+z^2+W^2)^{\left(\frac{1}{2}\right)}},\ x=-W..W\right),\ y=-W..W\right)$ *assuming* $z>0,\ W>0$;

$$-\frac{2\left(-\mathrm{I}\operatorname{arctanh}\left(\frac{W^2+\mathrm{I}\,W z+z^2}{W\sqrt{2\,W^2+z^2}}\right)+\mathrm{I}\operatorname{arctanh}\left(\frac{W^2-\mathrm{I}\,W z+z^2}{W\sqrt{2\,W^2+z^2}}\right)-\pi\right)}{z} \tag{4}$$

> $limit(\%,\ W=\infty)$;

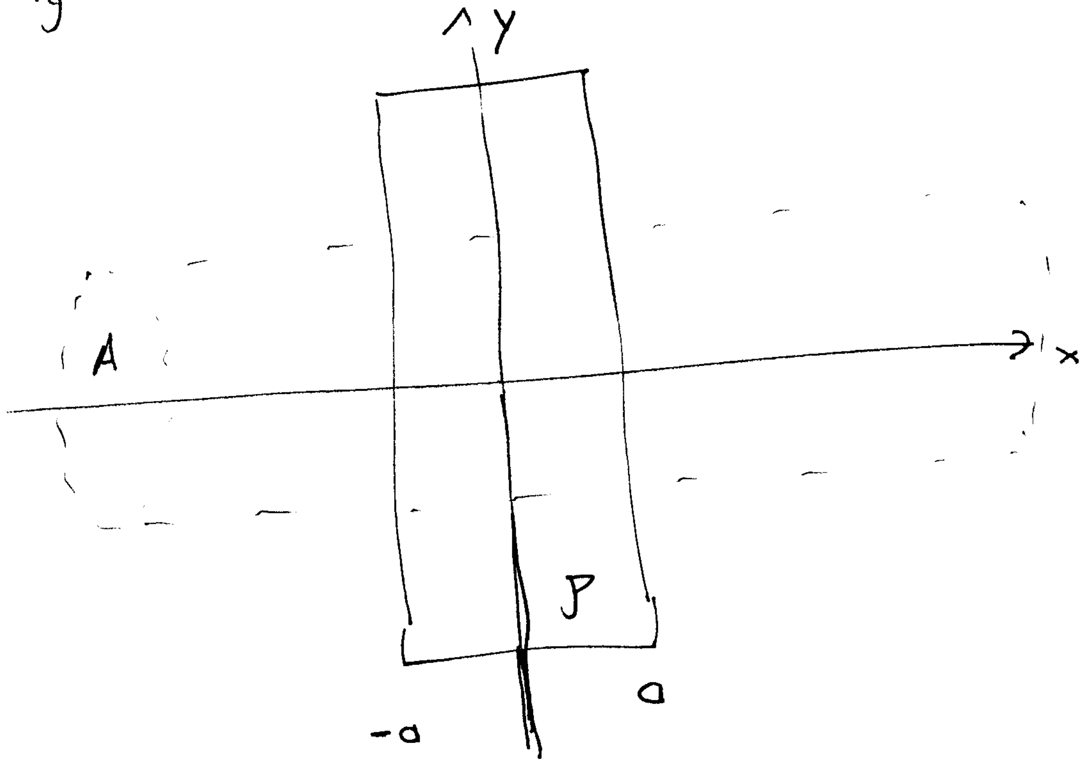
$$\frac{2\,\pi}{z} \tag{5}$$

>

Recall Gauss' law

$$\Phi = \frac{Q_{enc}}{\epsilon_0}$$

Consider a slab of volume charge centered at the origin and // to the y-z plane.



Calculate the field

By reflection symmetry $\vec{E}(0) = 0$.

Field points away from the slab for $x > a$ and $x < -a$.

By symmetry, the field a distance d from the slab has the same magnitude for $x = d$ and $x = -d$.

The flux out of the cylinder draws is $2|E_d|A = \Phi = \dfrac{Q_{enc}}{\varepsilon_0}$

The charge enclosed in the cylinder is

$$Q_{enc} = A(2a)\rho$$

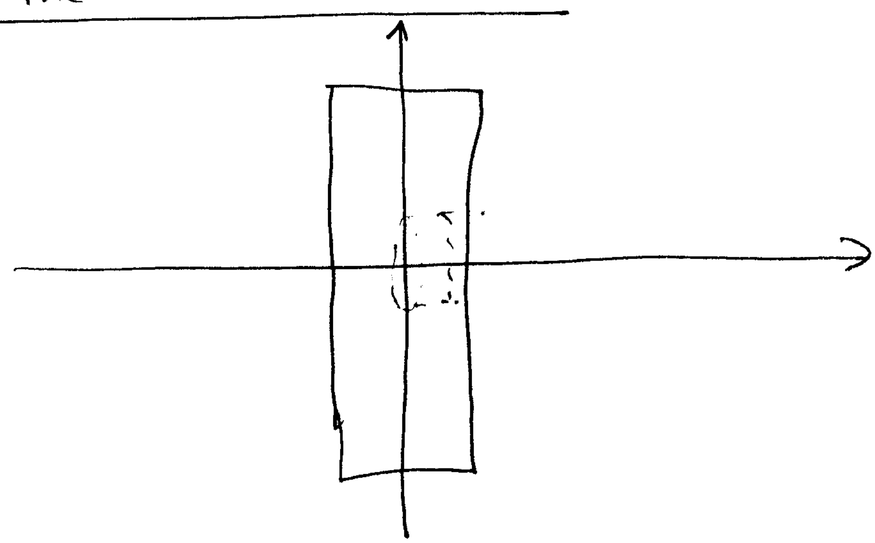
The field at d is the $|E_d| = \dfrac{Q_{enc}}{2A\varepsilon_0} = \dfrac{A(2a)\rho}{2A\varepsilon_0}$

$$= \frac{a\rho}{\varepsilon_0}$$

For $x < -a$, $\vec{E} = -\dfrac{a\rho}{\varepsilon_0}\hat{x}$

For $x > a$, $\vec{E} = \dfrac{a\rho}{\varepsilon_0}\hat{x}$

Calculate the field in the slab ($x>0$).



Place a Gaussian surface with left end at $x=0$ and right end at $x=x$.

The charge enclosed is $Q_{enc} = \rho V = \rho Ax$

The flux out of the surface is $E(x)A$. By Gauss' law, $\Phi = \frac{Q_{enc}}{\varepsilon_0}$

$$E(x)A = \frac{\rho Ax}{\varepsilon_0}$$

$$E(x) = \frac{\rho x}{\varepsilon_0}$$

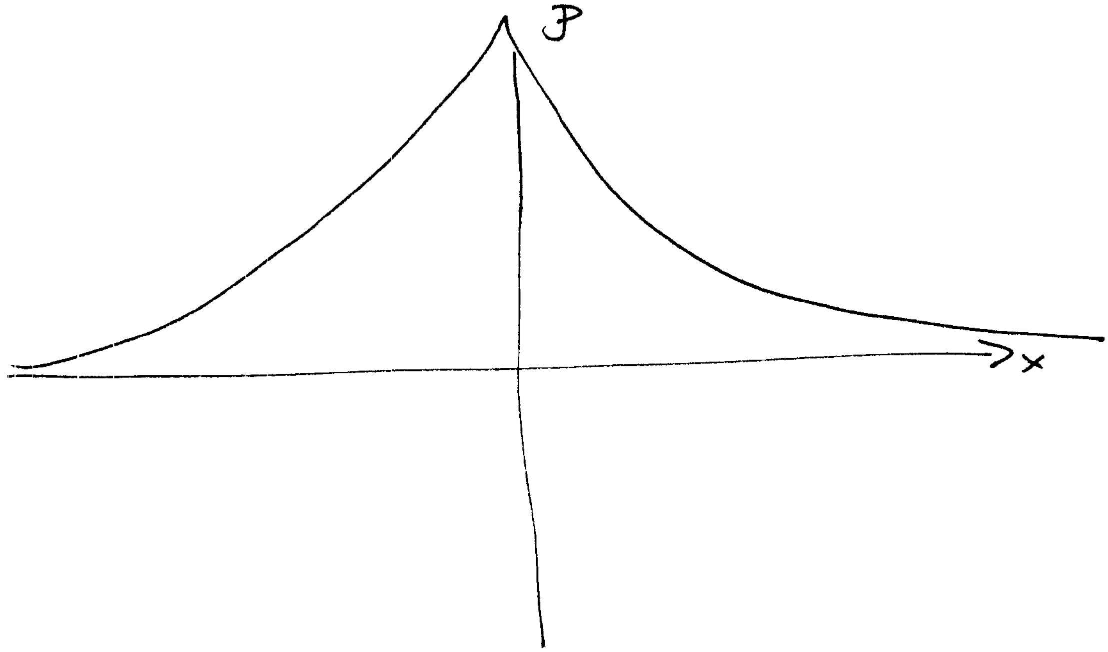
$$\vec{E}(x) = \frac{\rho x}{\varepsilon_0} \qquad x > 0 \qquad x < a$$

$$\vec{E}(x) = \frac{\rho x}{\varepsilon_0} \qquad x < 0, \ -a < x$$

by symmetry.

Consider the charge density

$$\rho(x) = \gamma e^{-a|x|}$$



Once again the system is symmetric, so $|E(x)| = |E(-x)|$

The flux out of the cylindrical Gaussian surface is now

$$2AE(x) = \frac{Q_{en}(x)}{\varepsilon_0}$$

$$Q_{enc} = \int_{-x}^{x} A\gamma e^{-a|x|}dx = 2\gamma A \int_0^x e^{-au}du$$

$$= -\frac{2\gamma A}{a} e^{-au}\Big|_0^x = \frac{2\gamma A}{a}\left(1 - e^{-ax}\right)$$

$$\vec{E} = \frac{\gamma}{a\,\varepsilon_0}\left(1 - e^{-ax}\right)\hat{x} \qquad x > 0$$

Total Charge of System

$$Q = \frac{2\gamma A}{a} \qquad \sigma = \frac{Q}{A} = \frac{2\gamma}{a}$$

<u>Infinite Plane</u> $\quad \dfrac{\sigma}{2\varepsilon_0} = \dfrac{\gamma}{a\,\varepsilon_0}$ ✓