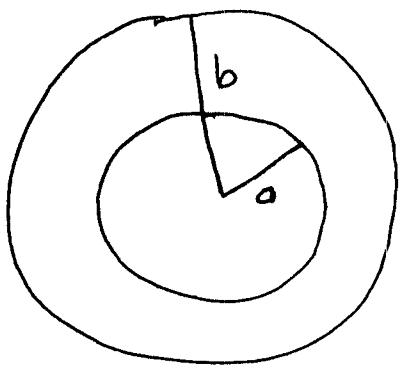
Electrostatic Pressure.

$$\vec{F}_{12} = \frac{\left(9\times10^{9}\frac{Nm^2}{c^2}\right)(3\times10^{-9}c)(5\times10^{-9}c)}{(\sqrt{34}\times10^{-2}m)^2}\left(\frac{-3}{\sqrt{34}}, \frac{5}{\sqrt{34}}, 0\right)$$

Now, consider two spherical shells of charge $\pm Q$

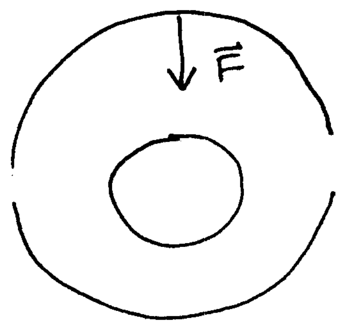


Compute force per unit area (pressure) the inner sphere exerts on outer sphere.

As you may recall (or will recall next lecture) the field is

$$\vec{E} = \begin{cases} 0 & r<a \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r} & a<r<b \\ 0 & r>b \end{cases}$$

Consider the force the inner sphere and the bottom half of the top sphere exert on the top half of the top sphere.



To get the force, we integrate the force per unit area over the top hemisphere.

$$\vec{F}_{top} = \int_{top} \frac{-Q^2}{32\pi^2\epsilon_0 b^2} \hat{r}\, da$$

$$= -P_0 \int_{top} \hat{r}\, da$$

$$da = (b\, d\phi)(b \sin\theta\, d\theta) = b^2 \sin\theta\, d\phi\, d\theta$$

$$\hat{r} = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$$

$$= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

The field exerts a force on both shells. The force per unit area (pressure) on the outer shell is

$$\frac{\vec{F}_b}{A} = \sigma_b \vec{E}(b)$$

where $\sigma_b = \dfrac{-Q}{4\pi b^2}$

But what is $\vec{E}(b)$? For $\vec{E}(b-\varepsilon) = \dfrac{Q}{4\pi\varepsilon_0 b^2}\hat{r}$

but $\vec{E}(b+\varepsilon) = 0$. We have to use the average of the two fields, $\vec{E}_{ave} = \dfrac{1}{2}\dfrac{Q}{4\pi\varepsilon_0 b^2}\hat{r}$

$$\frac{\vec{F}_b}{A} = \sigma_b \vec{E}_{ave} = \left(\frac{-Q}{4\pi b^2}\right)\left(\frac{Q}{8\pi\varepsilon_0 b^2}\right)\hat{r}$$

$$= \frac{-Q^2}{32\pi^2\varepsilon_0 b^4}\hat{r} \equiv -P_0\hat{r}$$

where P_0 is the pressure.

By symmetry, the total force on either the inner or outer shell is zero.

$$F_x = -P_0 \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi \,(b^2 \sin\theta)(\sin\theta\cos\phi) = 0$$

$$\int_0^{2\pi} d\phi \cos\phi = 0$$

Likewise $F_y = 0$.

$$F_z = -P_0 \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi \,(b^2 \sin\theta)(\cos\theta)$$

$$= -b^2 P_0 \left[\int_0^{\pi/2} d\theta \sin\theta\cos\theta\right]\left[\int_0^{2\pi} d\phi\right]$$

$$-b^2 P_0 \left[-\frac{1}{2}\cos^2\theta\Big|_0^{\pi/2}\right] \cdot \left[2\pi\right]$$

$$= -b^2 \pi P_0$$

$$\vec{F} = -b^2 \pi P_0 \,\hat{z} = -\frac{Q^2}{32\pi\varepsilon_0 b^2}\,\hat{z}$$

which is dimensionally correct.

Differential Gauss' Law

Ex Consider the field $\vec{E} = \gamma \rho \hat{\rho}$.

(a) Is this a possible static electric field?

(b) If so, compute the charge density.

(c) Find the potential.

If $\vec{E}$ is an electric field, then $\nabla \times \vec{E} = 0$

$$\nabla \times \vec{E} = \cancel{0\hat{\rho} + 0}$$

$$\left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \hat{\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \hat{\phi}$$

$$+ \frac{1}{\rho} \left(\frac{\partial (\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi} \right) \hat{z}$$

$$= 0 \checkmark$$

(b) $\nabla \cdot \vec{E} = \rho / \varepsilon_0$ (ρ is charge, let $\rho = \Gamma$)

$$\Gamma = \varepsilon_0 \nabla \cdot \vec{E}$$

$$= \varepsilon_0 \frac{1}{\rho} \frac{\partial(\rho A_\rho)}{\partial \rho} + 0 + 0$$

$$= \varepsilon_0 \frac{1}{\rho} \frac{\partial \Gamma \rho^2}{\partial \rho}$$

$$= 2 \varepsilon_0 \Gamma$$

Is this what we expected? Yes, last lecture we calculated the field inside a tube of charge as $\frac{\rho \Gamma}{2\varepsilon_0} \hat{\rho}$.

(c) $\vec{E} = -\nabla V = -\frac{\partial V}{\partial \rho} \hat{\rho} + 0 + 0$

$$V(\rho) = -\int \Gamma \rho \, d\rho = -\frac{\Gamma \rho^2}{2} + C$$

<u>Ex</u> Consider $\vec{E} = r\,\hat{\theta}$. Can this be an electric field

$\Rightarrow$ Spherical Implied

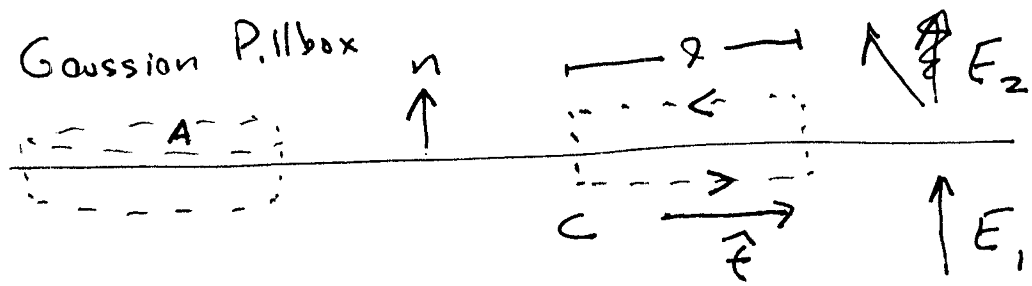
$$\nabla \times \vec{E} = \frac{1}{r\sin\theta}\left(-\frac{\partial r}{\partial \theta}\right)\hat{r}$$

$$+\ 0 \qquad + \left(\frac{1}{r}\,\frac{\partial\, r \cdot r}{\partial r} - 0\right)\hat{\phi}$$

$$= 2 \quad \neq 0$$ not a static electric field.

Electrostatic Boundary Conditions

Consider a sheet of charge or any interface where there might be different fields on the two sides of the interface.



Gauss' Law Apply GL to short cylinder with faces parallel to the interface

$$\Phi = (\vec{E}_2 \cdot \hat{n}) A - (\vec{E}_1 \cdot \hat{n}) A = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

Let $\vec{E}_2 \cdot \hat{n} \equiv$ Normal Component of the Field $\equiv E_{2n}$

At an interface, $E_{2n} - E_{1n} = \sigma/\varepsilon_0$ ALWAYS

Field is Irrotational $\nabla \times \vec{E} = 0 \Rightarrow \oint \vec{E} \cdot d\vec{l} = 0$

Let $\hat{t}$ be the tangent to the surface

$$\oint \vec{E} \cdot d\vec{l} = \vec{E}_1 \cdot \hat{t}\, l - \vec{E}_2 \cdot \hat{t}\, l = 0$$

$$\vec{E}_1 \cdot \hat{t} = \vec{E}_2 \cdot \hat{t}$$

$\Rightarrow$ For <u>STATIC</u> charge distributions, the tangential component of the electric field is continuous.

Gauss' Law Redux

Gauss' Law – Flux out of a closed surface

$\propto Q_{enc}$

$$\Phi = EA_s = \frac{Q_{enc}}{\varepsilon_0}$$

if we choose a Gaussian surface where $\vec{E} \perp \hat{n}$

or $\vec{E} \parallel \hat{n}$ and constant.

Spherical Systems – Gaussian surface is a sphere of radius r.

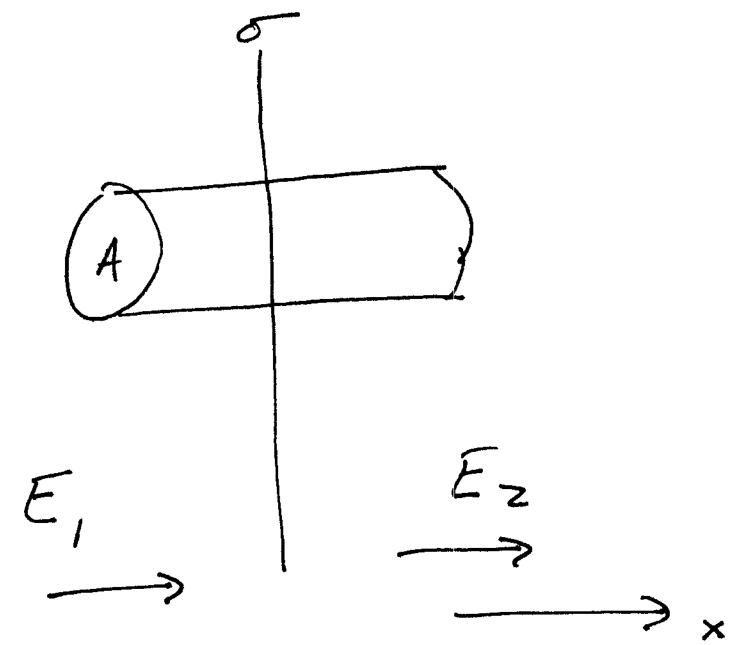
$$\Phi = EA_s = 4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_0}$$

$$\vec{E} = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}\hat{r}$$

Cylindrical Systems Gaussian surface is a co-axial cylinder with length L and radius ρ.

$$\Phi = 2\pi\rho L E = \frac{Q_{enc}}{\varepsilon_0} \qquad \vec{E} = \frac{Q_{enc}}{2\pi\varepsilon_0 \rho L}\hat{\rho}$$

<u>Planar Systems</u> - The Gaussian surface is a cylinder with end area A



Guess field directions,

$$\Phi = E_2 A + (-E_1 A) = \frac{Q_{enc}}{\epsilon_0}$$

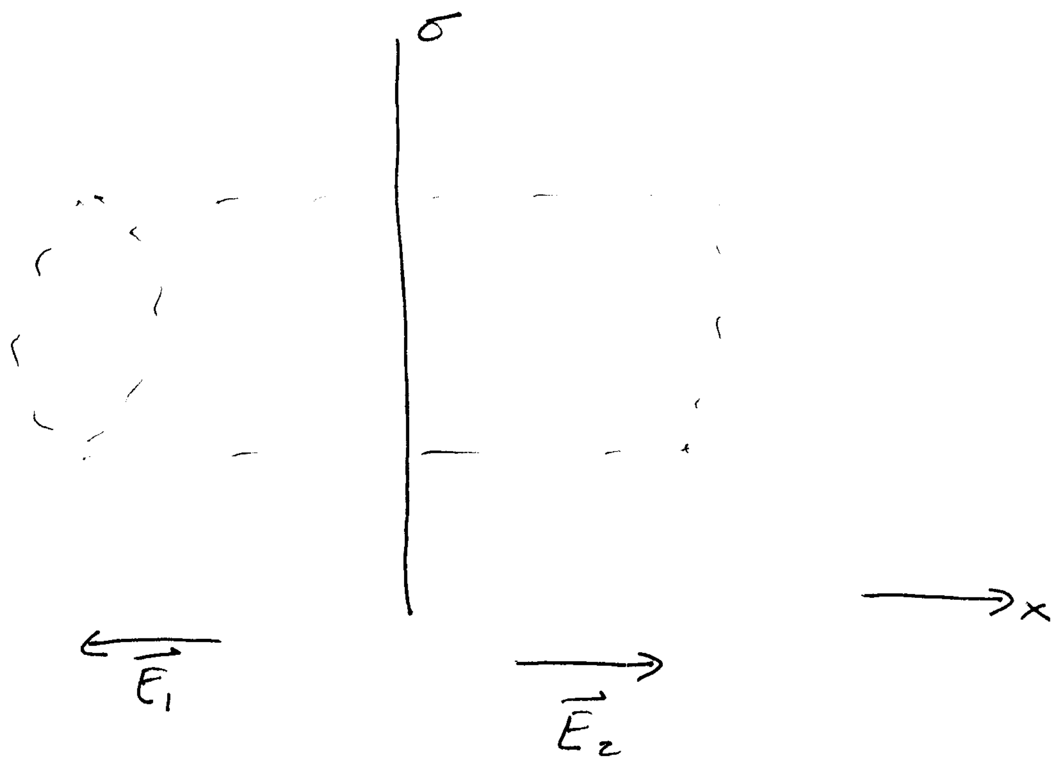
By reflection symmetry, $E_1 = -E_2$

$$2 E_2 A = \frac{Q_{enc}}{\epsilon_0} \qquad Q_{enc} = \sigma A$$

$$\vec{E}_2 = \frac{\sigma}{2\epsilon_0}\hat{x}$$

$$\vec{E}_1 = \frac{-\sigma}{2\epsilon_0}\hat{x}$$

Let's Guess Differently and Correctly



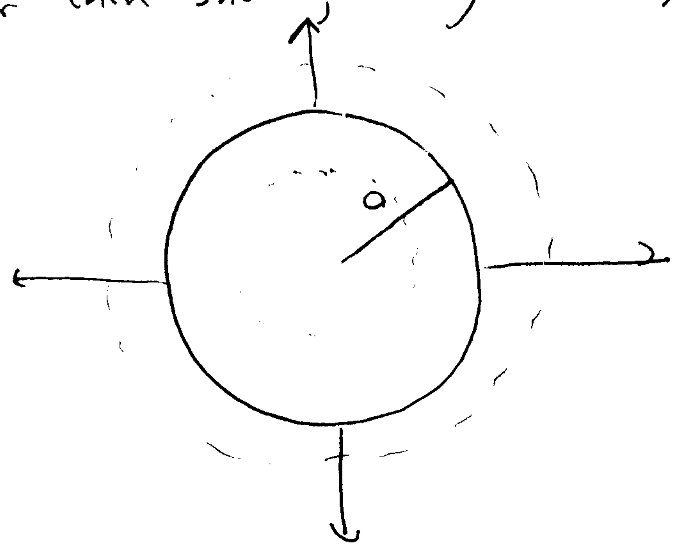
$$\Phi = E_1 A + E_2 A = \frac{\sigma A}{\varepsilon_0} \qquad \vec{E}_1 = \vec{E}_2$$

$$2E_2 = \frac{\sigma}{\varepsilon_0}$$

$$\vec{E}_2 = \frac{\sigma}{2\varepsilon_0}\hat{x}$$

Spherical Symmetry

Consider thin shell, charge density σ and radius a.



r < a $\quad Q_{enc} = 0$

$$\Phi = 4\pi r^2 E = 0 \Rightarrow \vec{E} = 0$$

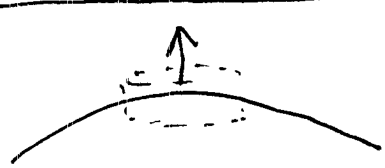
$\Rightarrow$ Inside all charge, field is zero.

r > a $\quad Q_{enc} = 4\pi a^2 \sigma$

$$\Phi = E A_s = E 4\pi r^2 = 4\pi a^2 \sigma \equiv Q$$

$$\vec{E} = \frac{4\pi a^2 \sigma}{4\pi r^2 \varepsilon_0}\hat{r} = \frac{Q}{4\pi \varepsilon_0 r^2}\hat{r}$$

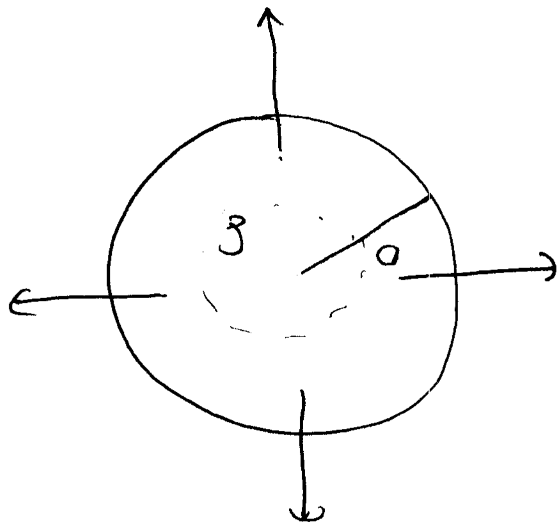
Check with Pillbox



$$\Phi = \vec{E}_+(b)A - 0A = \frac{\sigma A}{\varepsilon_0}$$

$$= \frac{4\pi a^2 \sigma A}{4\pi a^2 \varepsilon_0} = \frac{\sigma A}{\varepsilon_0} \Rightarrow \sigma = \sigma$$

Volume Charge ρ, radius a



$\underline{r<a}$ $\quad Q_{enc} = \frac{4}{3}\pi r^3 \rho$ $\qquad \vec{E} = \frac{4/3\,\pi r^3 \rho}{4\pi\varepsilon_0 r^2}\hat{r}$

$$= \frac{r\rho}{3\varepsilon_0}\hat{r}$$

$\underline{r>a}$ $\quad Q_{enc} = \frac{4}{3}\pi a^3 \rho$ $\qquad \vec{E} = \frac{\frac{4}{3}\pi a^3 \rho}{4\pi\varepsilon_0 r^2}\hat{r}$

How about a non-uniform volume charge $\rho(r) = \gamma r^2$, radius a.

Total charge $\quad Q = \int \rho(r)\, dv$

$$= \int_0^{\pi} d\theta \cancel{\sin} \int_0^{2\pi} d\phi \int_0^{a} dr \left(r^2 \sin\theta\right) \gamma r^2$$

$$dv = \underbrace{dr}\,\underbrace{r\,d\phi}\,\underbrace{r\sin\theta\, d\theta}$$

<u>Angle bits</u>

$$\underbrace{\int_0^\pi d\theta \sin\theta}_{\underbrace{-\cos\theta\Big|_0^\pi}_{2}} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} = 4\pi$$

$$Q = \int_0^a (4\pi r^2)\,\rho(r)\,dr = 4\pi\gamma \int_0^a r^4 dr$$

$$= \frac{4\pi\gamma}{5} a^5$$

<u>$r > a$</u>

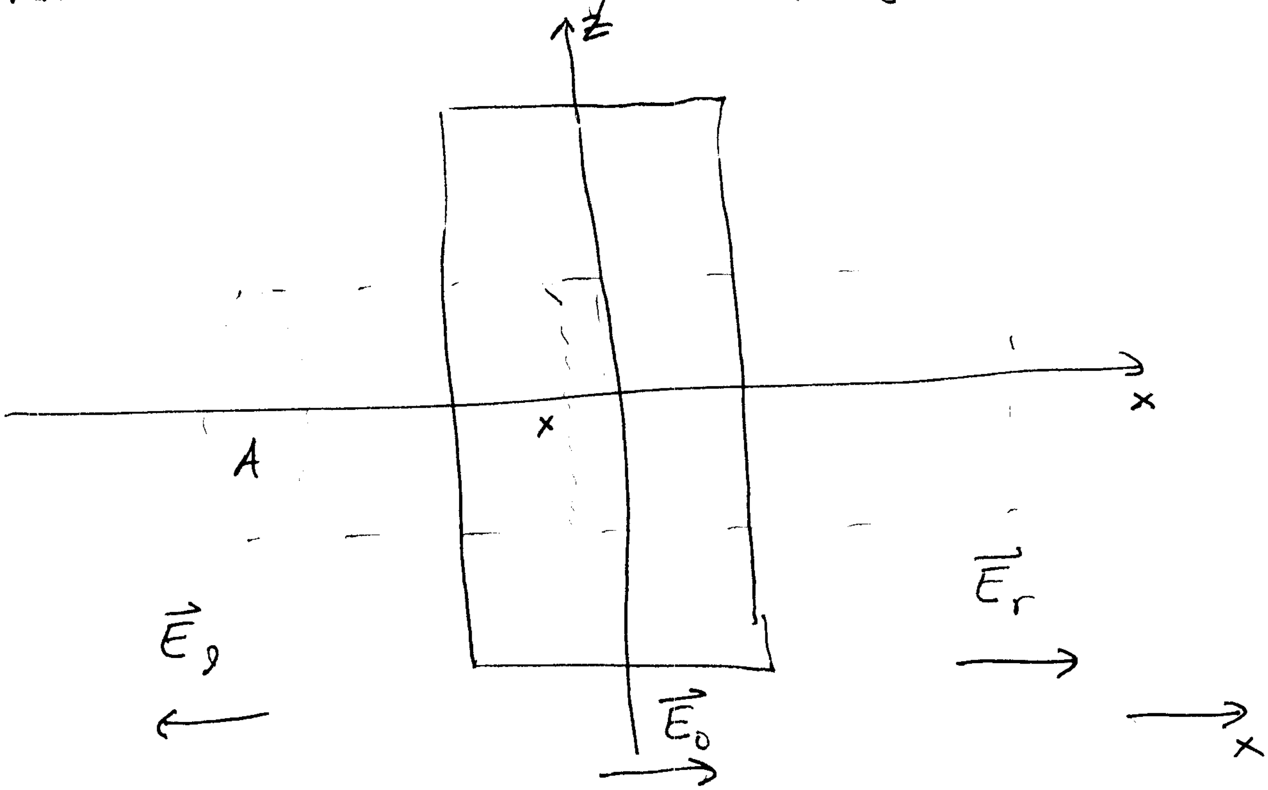
$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{4\pi\gamma a^5/5}{4\pi\varepsilon_0 r^2}\hat{r}$$

$r < a$

$$Q_{enc} = \frac{4\pi\gamma r^5}{5} \qquad \vec{E} = \frac{4\pi\gamma r^5/5}{4\pi\varepsilon_0 r^2}\hat{r}$$

$$= \frac{\gamma r^3}{5\varepsilon_0}\hat{r}$$

Slab with uniform charge density ρ from $-a$ to a.



$$\Phi = E_l A + E_r A = \frac{Q_{enc}}{\varepsilon_0} = \frac{A(2a)\rho}{\varepsilon_0}$$

$E_l = E_r$ (reflection) $\Rightarrow$ $\vec{E}_l = -\vec{E}_r$

$$2E_r = \frac{2a\rho}{\varepsilon_0}$$

$$\vec{E}_r = \frac{a\rho}{\varepsilon_0}\hat{x} \qquad \vec{E}_l = -\frac{a\rho}{\varepsilon_0}\hat{x}$$

$\underline{-a<x<a}$

Let x be the location of edge of a second Gaussian surface in the slab.

$$Q_{enc} = \left(x - \left(-a\right)\right)A\rho = \left(x + a\right)A\rho$$

$$\Phi_{\phi} = E_{\rho} A + E_0 A = \frac{(x + a/2) A \rho}{\varepsilon_0}$$

$$E_0 = \frac{(x + a/2)\rho}{\varepsilon_0} - \frac{a\rho}{\varepsilon_0}$$

$$E_0 = \frac{x\rho}{\varepsilon_0}$$

$$\vec{E}_0 = \frac{x\rho}{\varepsilon_0}\hat{x}$$