Electromotive Force

Consider a closed path C, the total work done on a charge Q as it traverses the path is

$$\text{Work} = \oint_C \left(\vec{F}_e + \vec{F}_m\right) \cdot d\vec{l}$$

$$= q \oint_C \left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot d\vec{l}$$

Electromotive force (emf or $\mathcal{E}$) is the work per unit charge done by the electric and magnetic field as a charge q moves along the path C

$$\text{emf} = \oint_C \left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot d\vec{l}$$

<u>Motional EMF</u> - The part of the emf that results from the motion of the path C

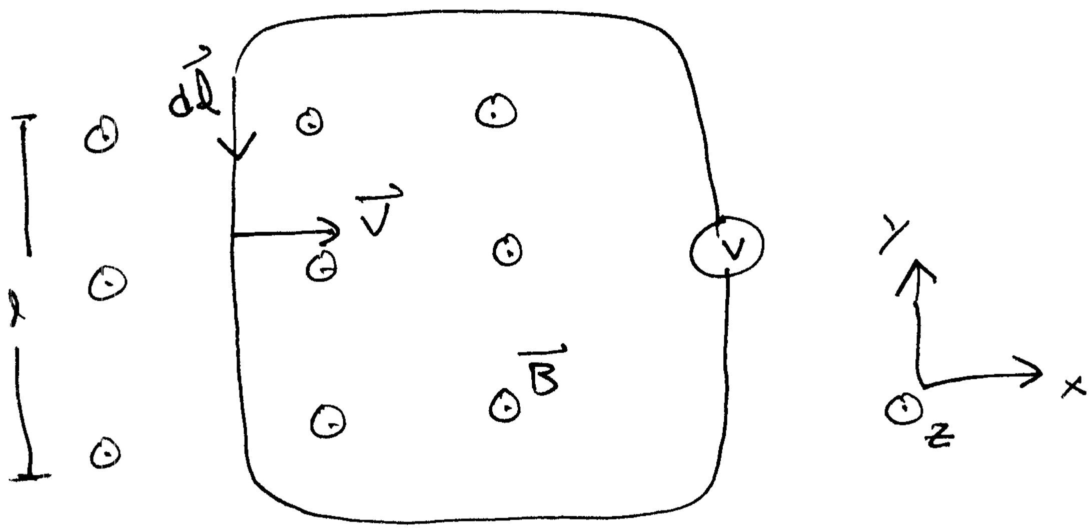
$$\text{motional emf} = \oint_C \left(\vec{v} \times \vec{B}\right) \cdot d\vec{l}$$

where $\vec{v}$ is the velocity of the path C.

⇒ If the path C is through a conductor the emf will appear as a voltage on a voltmeter

⇒ For static fields, $\oint \vec{E}\cdot d\vec{l} = 0$

Consider the following experiment, pull a wire out of a magnet



No electric field is present, so

$$\text{emf} = \oint_C \vec{v}\times\vec{B}\cdot d\vec{l}$$

$\vec{B} = B_0\hat{z}$ $\quad\quad$ $\vec{v} = v_0\hat{x}$ $\quad\quad$ $d\vec{l} = -\hat{y}\,dy$

$$\vec{v}\times\vec{B} = v_0 B_0(\hat{z}\times\hat{x}) = +\hat{y}\, v_0 B_0$$

$$\vec{v}\times\vec{B}\cdot d\vec{l} = -v_0 B_0\, dy$$

$$\text{emf} = \oint_C (\vec{v} \times \vec{B}) \cdot d\vec{l}$$

$$= -v_0 B_0 \int dy = -v_0 B_0 l$$

where the negative sign indicates that current will flow opposite the direction of $d\vec{l}$

⟹ Play with this some, note that is only directions of motion where the wire cuts through field lines that produce emf.

⟹ But cutting through field lines implies it is only motions of the wire that change the magnetic flux that produce emf

<u>Magnetic flux</u> (Φ_m) - The magnetic flux through surface S is

$$\Phi_m = \int_S \vec{B} \cdot d\vec{a}$$

⟹ If field uniform and loop flat with normal $\hat{n}$

$$\Phi_m = (\vec{B} \cdot \hat{n}) A$$

where A is the surface area.

<u>Ex</u> Let's put some numbers in to get a feel for the size of the motional emf

Let $B_0 = \frac{1}{4}\,T$

$V_0 = 10\,m/s$

$l = 10\,cm$

$$emf = V_0 B_0 l = \frac{1}{4}\,V$$

Not much but observable.

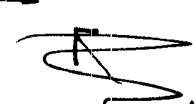
Multiple Turns (N) - Our definition of flux is general, we will meet will be but many of the surfaces such a would of multiple turns, the flux through the single turn circuit. surface will be N times

$$\Phi_m = N\Phi_0 = N \oint_S \vec{B}$$

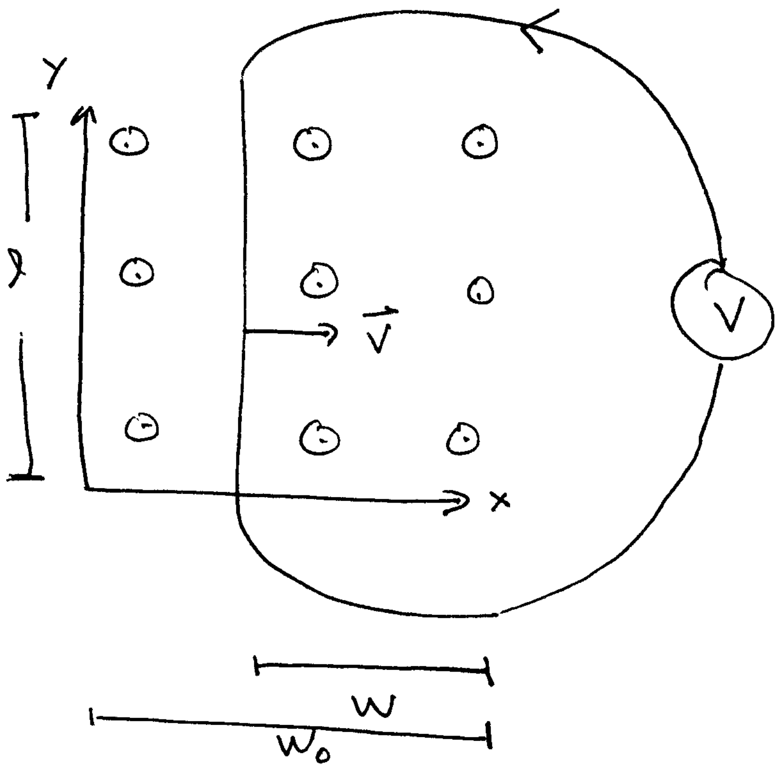
Let's write the motional emf in terms of the change in flux.

$$emf = -V_0 B_0 l$$

~~$\Phi_m = B_0 l x$~~



Toroid



$$\Phi_m = B_o w l$$

$$= B_o (W_o - x) l$$

$$\dot{\Phi}_m = \frac{d\Phi_m}{dT} = -B_o W_o l$$

$$\text{emf} = -\dot{\Phi}_m$$

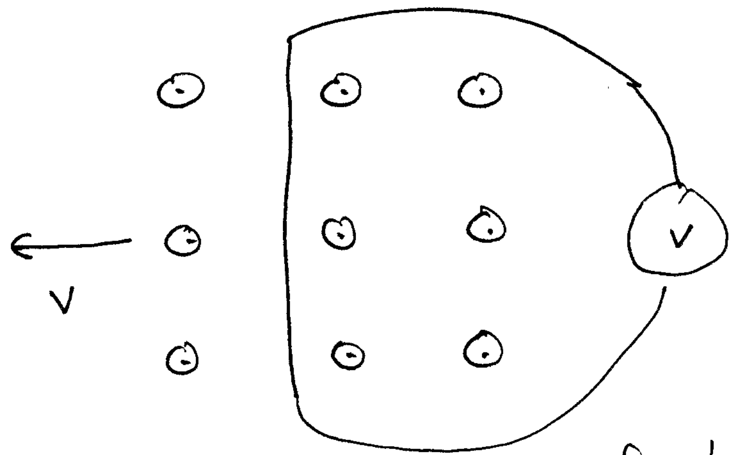
(we have a sign flopping around)

<u>Flux Rule</u> - The emf around a closed curve C is proportional to change in flux through loop

$$\text{emf} = -\dot{\Phi}_m = -\frac{d\Phi_m}{dT}$$

$\Rightarrow$ Applies to moving loops and stationary loops.

Now consider the complementary experiment



move the magnet. By the principle of relativity, the meter must read the same.

$$\text{emf} = \oint_C (\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{l}$$

but C is not moving, so $\vec{v} = 0$ and $\vec{v} \times \vec{B} = 0$.

$$\Rightarrow \oint_C \vec{E} \cdot d\vec{l} \neq 0$$

$\Rightarrow$ Changing magnetic field creates electric field.

$\Rightarrow$ Our flux analysis is still good, in fact the change in flux we calculate is exactly the same.

We have a sign running around that's problematic.

<u>Lenz' Law</u> - The induced current flows in a direction s.t. the flux generated by the current <u>opposes</u> the change in flux.

$\Rightarrow$ Induced current is related to emf by

$$\text{emf} = \int_C \vec{E} \cdot d\vec{l}$$

, if emf > 0, $\vec{E}$ points in direction of current.

Let's find Faraday's law. Consider the special case of the flux rule where C is stationary.

$$\text{emf} = \oint_C \vec{E} \cdot d\vec{l} = -\frac{d\Phi_m}{dT} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{a}$$

use Stoke's thm

$$\oint_C \vec{E} \cdot d\vec{l} = \int_S (\nabla \times \vec{E}) \cdot d\vec{a}$$

$$= -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{a}$$

because S is stationary this becomes

$$\int_S (\nabla \times \vec{E}) \cdot d\vec{a} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$$

Since this applies to all S, $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

<u>Faraday's Law</u> – The integral of the electric field around a closed curve C is proportional to the time rate of change of the flux through the surface S bounded by C

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{a}$$

– or –

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$\Rightarrow$ Normal right handed sign convention applies.

$\Rightarrow$ Absolutely true at all points in space.

$\Rightarrow$ Integral form is special case of the flux rule.

Where did the work come from to accelerate the current?

You.

Before I do a bunch of deforming loops, let's play with the fields.

Compare Ampere + Faraday with $\vec{J}=0$

Ampere

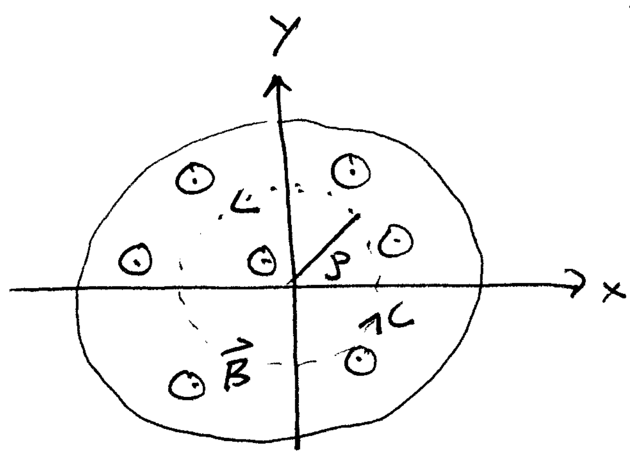
$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{d}{dt} \int \vec{E}\cdot d\vec{a}$$

Faraday $\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a}$

Except for constants and a minus sign, they are the same with the fields exchanged. This means all of our Ampere's Law tricks work with Faraday.

Ex Consider a cylindrical region $\rho < a$ containing an increasing magnetic field out of the page

$$\vec{B} = B_0 \frac{t}{\tau} \hat{z}$$



Use a "Faradian" Path of radius ρ. Select the direction of the curve so the normal points in positive z direction.

$$\Phi_m = \int_S \vec{B} \cdot d\vec{a}$$

$$= \pi \rho^2 \frac{B_0 t}{T}$$

$$\dot{\Phi}_m = \frac{\pi \rho^2 B_0}{T}$$

By symmetry, (rotation about axis) $\vec{E}$ must be circular about axis $\Rightarrow d\vec{l} \parallel \vec{E}$

and $\vec{E} \cdot d\vec{l} = E\, dl$

$$\oint_C \vec{E} \cdot d\vec{l} = 2\pi\rho\, E(\rho)$$

$$= -\dot{\Phi}_m = -\frac{\pi \rho^2 B_0}{T}$$

$$\vec{E}(\rho) = -\frac{\rho B_0}{2T} \hat{\phi}$$

I used $\hat{\phi}$ as the direction because $\hat{\phi} \parallel d\vec{l}$

Check sign with Lenz' Law

Φ_m increasing out of page
emf would (if circuit existed) drive current ~~counter~~clockwise to oppose change in flux.
$-\hat{\phi}$ is clockwise ✓

Can we get the same thing from the differential form?

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{B_0}{\tau}\hat{z}$$

Look up $\hat{z}$ curl component in cylindrical symmetry.

$$(\nabla \times \vec{E})_z = \frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right)$$

$$\frac{\partial A_\rho}{\partial \phi} = 0 \quad \text{by cylindrical symmetry}$$

$$\frac{1}{\rho}\frac{\partial(\rho A_\phi)}{\partial \rho} = -\frac{B_0}{\tau}$$

$$\frac{\partial}{\partial \rho}(\rho A_\phi) = -\frac{B_0 \rho}{\tau}$$

$$\rho A_\phi = -\frac{B_0 \rho^2}{2\tau} + C$$

$$A_\phi = -\frac{B_0 \rho}{2\tau} + \frac{C}{\rho} \Rightarrow C = 0 \text{ for } A_\phi \text{ finite}$$

$$A_\phi = -\frac{B_0 \rho}{2\tau}$$

$$\vec{E} = -\frac{B_0 \rho}{2\tau} \hat{\phi}$$

Naturally, we would have to check this left the other components zero.