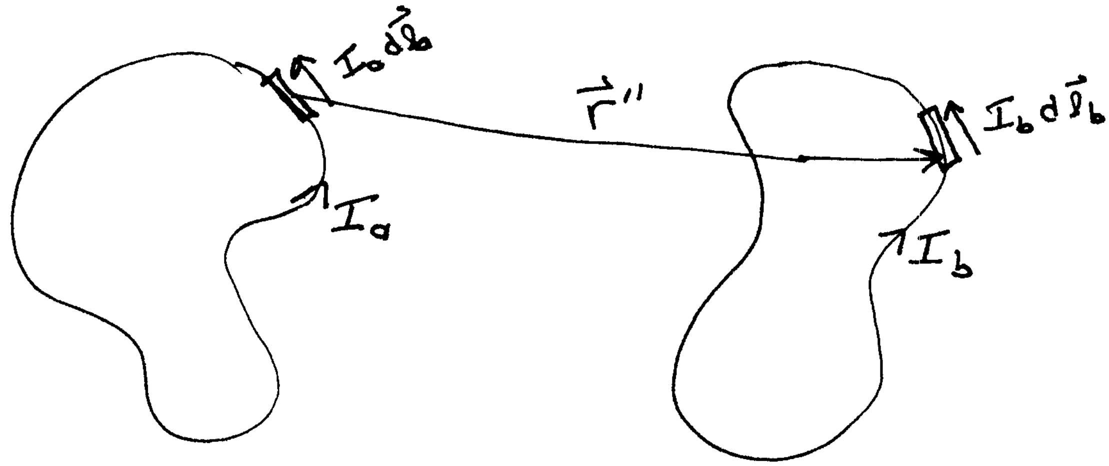
Magnetostatics

If you run current through one circuit, a force will be exerted on a second current carrying circuit.



The total force circuit a exerts on circuit b is

$$\vec{F}_{ab} = \frac{\mu_0}{4\pi} I_a I_b \oint_a \oint_b d\vec{l}_b \times \frac{d\vec{l}_a \times \hat{r}''}{r''^2}$$

where $\vec{r}''$ is the displacement vector from the infinitesimal current element $I_a d\vec{l}_a$ to the " " " $I_b d\vec{l}_b$

⇒ μ_0 = Permeability of Free Space
= $4\pi \times 10^{-7}$ Tm/A

⇒ Applies in quasi-state situations, situations where things do not change quickly.

As we did with Coulomb's force law, we will split the expression into a field and let the field exert the force.

<u>Biot-Savart Law</u> (static magnetic fields) -

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \oint_c \frac{d\vec{\lambda}' \times \hat{r}''}{r''^2}$$

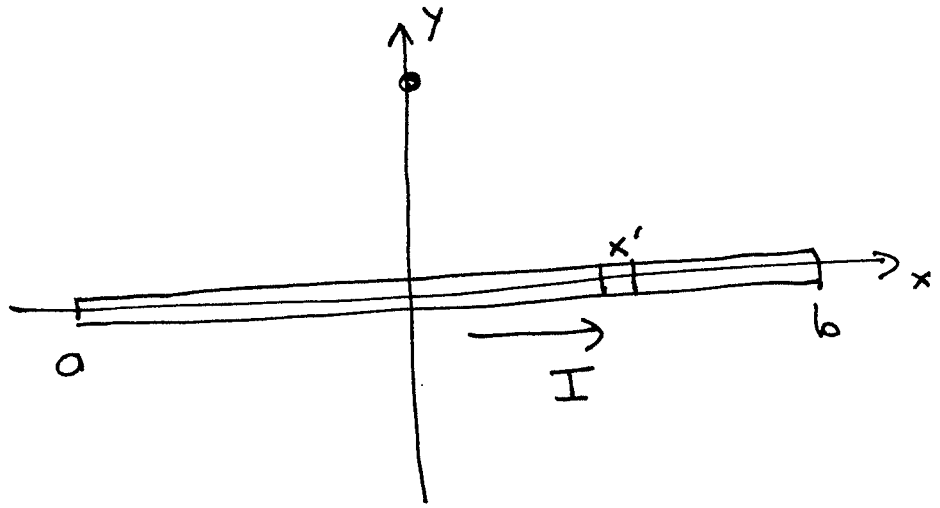
$$\vec{r}'' = \vec{r} - \vec{r}'$$

field location   source location

<u>Magnetic Force</u>

$$\vec{F}(\vec{r}) = I \oint_c d\vec{\ell}' \times B(\vec{r}')$$

Compute field of a straight wire segment



Symmetric Case First $\underline{a=-b}$, field at $(0,y,0)$

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int_C \frac{d\vec{l}' \times \hat{r}''}{|\vec{r}''|^2}$$

$$= \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}' \times \vec{r}''}{r''^3}$$

$$\vec{r}'' = \vec{r} - \vec{r}' = (0,y,0) - (x',0,0)$$

$$= (-x', y, 0)$$

$$d\vec{l}' = \hat{x}\, dx'$$

$$d\vec{l}' \times \vec{r}'' = (\hat{x}\, dx') \times (-x'\hat{x} + y\hat{y})$$

$$= y\, dx'\, \hat{x} \times \hat{y} = y\, dx'\, \hat{z}$$

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int_{-b}^{b}\frac{y\,dx'\,\hat{z}}{(x'^2+y^2)^{3/2}}$$

$$= \frac{\mu_0 I y\,\hat{z}}{4\pi}\int_{-b}^{b}\frac{dx'}{(x'^2+y^2)^{3/2}}$$

$$= \frac{\mu_0 I y\,\hat{z}}{4\pi}\left(\frac{2b}{y^2\sqrt{y^2+b^2}}\right)$$

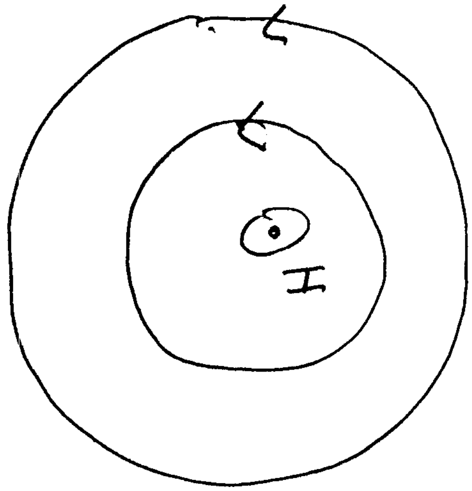
$$= \frac{\mu_0 I}{2\pi y}\left(\frac{b}{\sqrt{y^2+b^2}}\right)\hat{z}$$

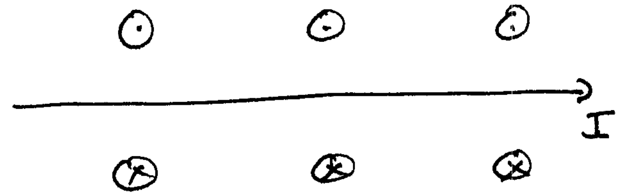
If $b \to \infty$, the wire becomes infinitely long.
Evidently, the field forms a circle around the wire.

<u>Field of an Infinite Straight Wire</u>

$$|B| = \frac{\mu_0 I}{2\pi R}$$

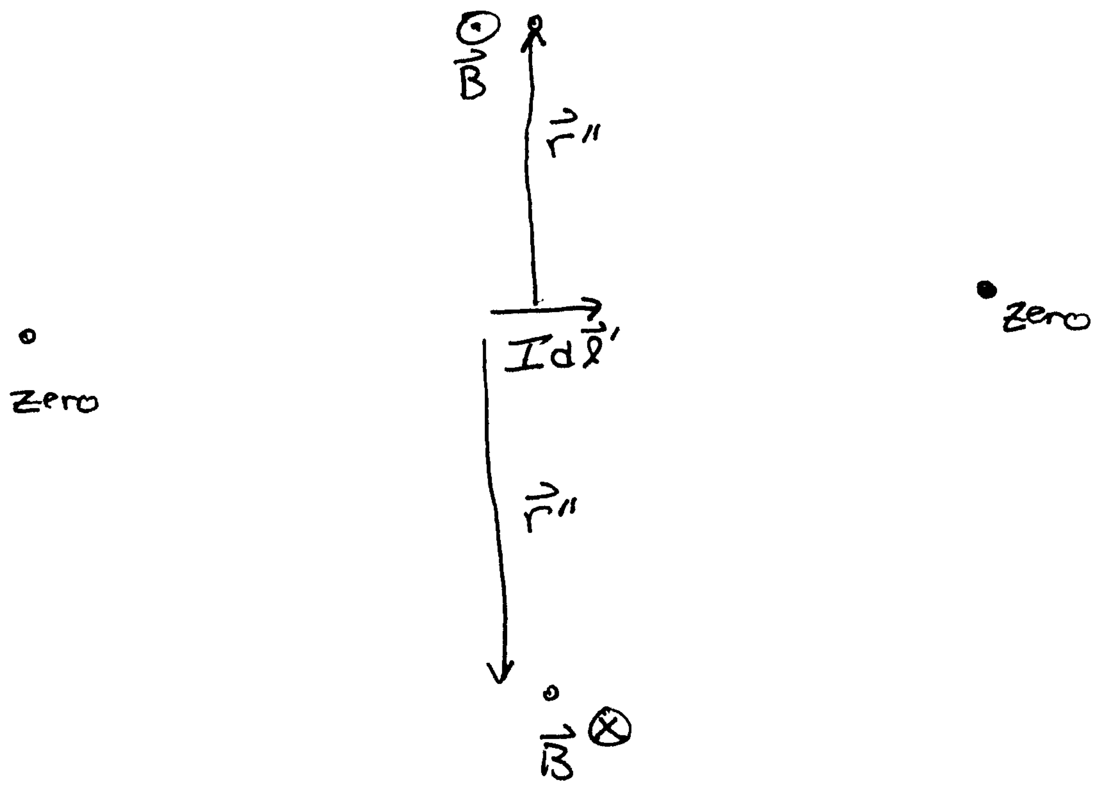
- RHR Wire - Grab wire with right hand with thumb in direction of current, fingers curl in direction of field.





How can we see this from the Biot-Savart law?

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l}' \times \vec{r}''}{r''^3}$$



$$> \; int\left(\frac{1}{\left(y^2+x^2\right)^{\left(\frac{3}{2}\right)}},\; x\right);$$

$$\frac{x}{\sqrt{y^2+x^2}\; y^2} \tag{1}$$

$$> \; int\left(\frac{1}{\left(y^2+x^2\right)^{\left(\frac{3}{2}\right)}},\; x=-a..a\right) \; assuming \;\; y>0;$$

$$\frac{2\,a}{\sqrt{y^2+a^2}\; y^2} \tag{2}$$

$>$

How much field are we dealing with?

<u>Ex</u> 1 cm for 20A wire.

$$B = \frac{\mu_0 I}{2\pi R} = \frac{\left(4\pi\times10^{-7}\,\frac{Tm}{A}\right)(20)}{2\pi(0.01m)}$$

$$= 4\times10^{-4}\,T$$

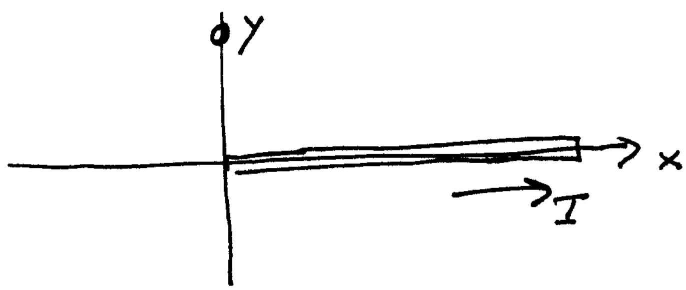
The Earth's field is $0.4\times10^{-4}\,T = 0.4$ Gauss.

It is no problem to buy 1T surface magnitization magnets so this in not much.

<u>Non-symmetric case</u>

$$\vec{B} = \frac{\mu_0 I}{4\pi y}\left(\frac{b}{\sqrt{y^2+b^2}} - \frac{a}{\sqrt{y^2+a^2}}\right)\hat{z}$$

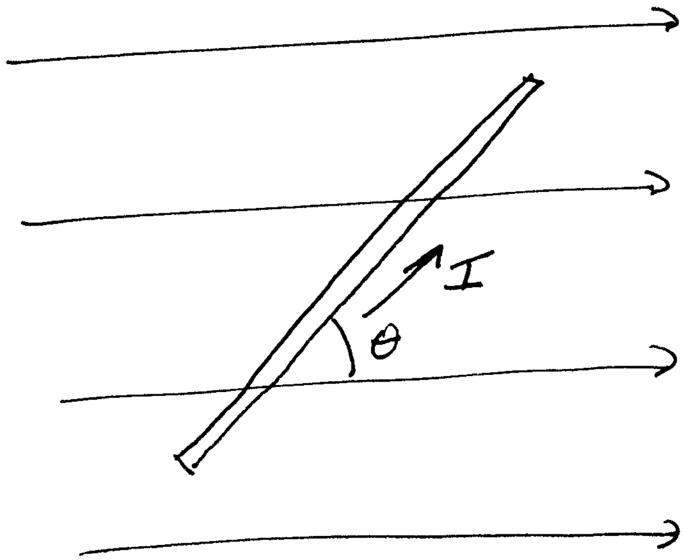
<u>Ex</u> Field of half an infinite wire $a = 0,\ b\to\infty$



$$\vec{B} = \frac{\mu_0 I}{4\pi y}\hat{z}$$

Now that we can make some field, let it exert a force.

<u>Ex</u> Force on a straight wire at an angle θ to uniform magnetic field $\vec{B} = B_0 \hat{x}$. Let L be the length of the wire.



$$\vec{F} = I \int_{\text{wire}} d\vec{l}' \times \vec{B}(\vec{r}')$$

The direction is into the page using the right hand rule (RHR).

$$\left| d\vec{l}' \times \vec{B}(\vec{r}') \right| = |d\vec{l}'| |\vec{B}| \sin\theta$$

$$= dl' B_0 \sin\theta$$

$$F = I \int_{\text{wire}} dl' B_0 \sin\theta = I B_0 \sin\theta \int dl' = I B_0 L \sin\theta$$

Let's do the crappy math for real. We are doing a line integral. Let $x \in [0, L\cos\theta]$ parameterize the line.

$$y = \tan\theta \, x$$

$$dy = \tan\theta \, dx$$

$$d\vec{l}' = dx'\hat{x} + dy'\hat{y}$$

$$= dx'\hat{x} + \tan\theta \, dx'\hat{y}$$

$$d\vec{l}' \times \vec{B} = (dx'\hat{x} + \tan\theta \, dx'\hat{y}) \times B_0\hat{x}$$

$$= B_0 \tan\theta \, dx' \hat{y} \times \hat{x}$$

$$= -B_0 \tan\theta \, dx' \, \hat{z}$$

$$\vec{F} = I \int_{wire} d\vec{l}' \times \vec{B} = -B_0 \tan\theta \, \hat{z} I \int_0^{L\cos\theta} dx'$$

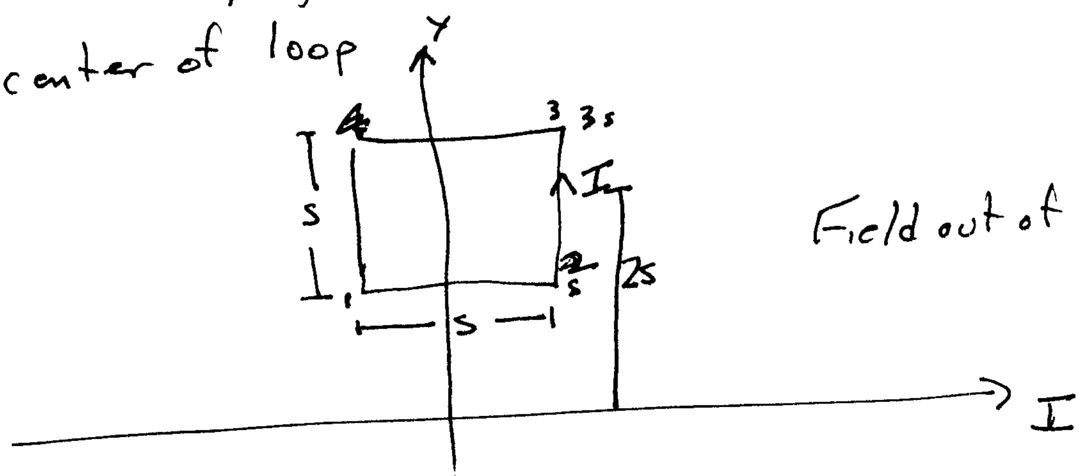
$$= -B_0 I \hat{z} \tan\theta \cdot L\cos\theta = -B_0 I L \hat{z} \sin\theta$$

Force on Straight Wire at Angle θ to field

$$|\vec{F}| = I B_0 L \sin\theta$$

$$\vec{F} = I\vec{L}\times\vec{B}$$

Ex Compute the force exerted on a loop of wire carrying current I_ℓ by an infinite wire carrying current I_w a distance $2s$ from center of loop



Field out of page

The magnetic field points out of the page at all points around the loop. Let $1 \to 2 \to 3 \to 4$ be corners of loop.

By RHR,

$1 \to 2$ Force $-\hat{y}$

$2 \to 3$ Force $+\hat{x}$

$3 \to 4$ Force $+\hat{y}$

$4 \to 1$ Force $-\hat{x}$

Evidently, the forces from $2 \to 3$ and $4 \to 1$ cancel.

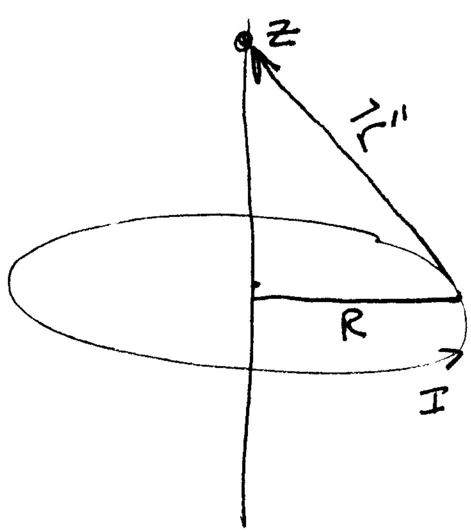
The total force is then

$$\vec{F} = I_\ell L \hat{x} \times \vec{B}(s) + I_\ell L (-\hat{x}) \times \vec{B}(3s)$$

$$= \left(\frac{I_\ell L \mu_0 I_w}{2\pi s}\right)(-\hat{y}) + \left(\frac{I_\ell L \mu_0 I_w}{6\pi s}\right)(\hat{y})$$

$$= -\frac{\mu_0 I_w I_\ell L}{3\pi s}\hat{y}$$

Let's examine another system, a circular loop of wire carrying current I.



$$d\vec{l}' = R\hat{\phi}\,d\phi$$

$$\vec{r}' = (R\cos\phi,\ R\sin\phi,\ 0) = R\hat{\rho}$$

$$\vec{r} = (0,\ 0,\ z) = z\hat{z}$$

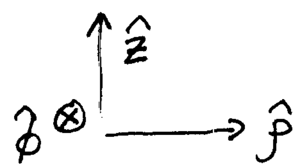
$$\vec{r}'' = \vec{r} - \vec{r}' = z\hat{z} - R\hat{\rho}$$

$$d\vec{l}' \times \vec{r}'' = R\,d\phi\,\hat{\phi} \times (z\hat{z} - R\hat{\rho})$$

$$= Rz\,d\phi\,(\hat{\phi}\times\hat{z}) - R^2 d\phi\,\hat{\phi}\times\hat{\rho}$$

Our right handed triple is $\hat{\rho}\times\hat{\phi} = \hat{z}$

$\hat{\phi}\times\hat{\rho} = -\hat{z}$ $\qquad \hat{\phi}\times\hat{z} = \hat{\rho}$



$$r'' = \sqrt{z^2 + R^2}$$

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{d\vec{l}' \times \vec{r}''}{r''^3} =$$

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \hat{\rho}\frac{Rz\,d\phi}{(z^2+R^2)^{3/2}} + \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{R^2 d\phi\,\hat{z}}{(z^2+R^2)^{3/2}}$$

$$= \frac{\mu_0 I R z}{4\pi(z^2+R^2)^{3/2}}\underbrace{\int_0^{2\pi} d\phi\,\hat{\rho}}_{0} + \frac{\mu_0 I R^2\,\hat{z}}{4\pi(z^2+R^2)^{3/2}}\underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$\vec{B} = \left(\frac{\mu_0 I}{2}\right)\left(\frac{R^2}{(z^2+R^2)^{3/2}}\right)\hat{z}$$

At center of ring with $z=0$ $\qquad \vec{B} = \frac{\mu_0 I}{2R}\hat{z}$