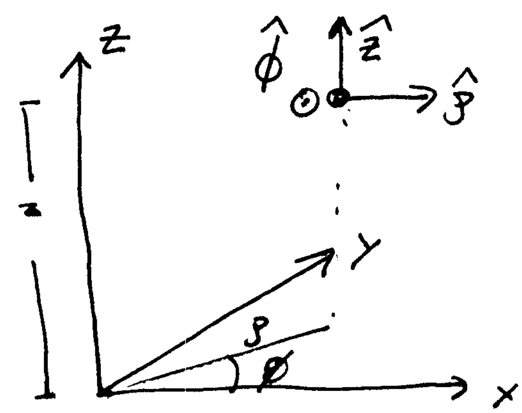
Curvilinear Coordinates

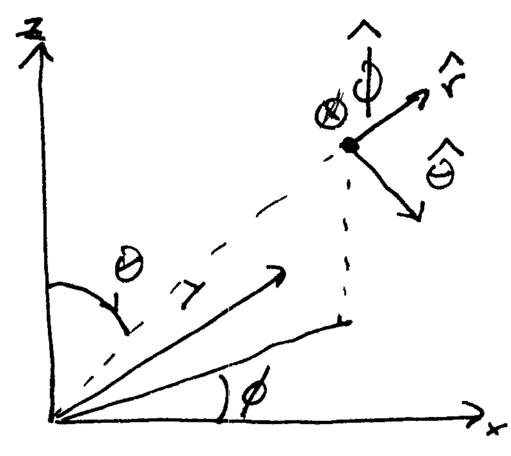
Cartesian $\hat{x} \times \hat{y} = \hat{z}$

Cylindrical $\hat{\rho} \times \hat{\phi} = \hat{z}$



Spherical

~~$\hat{r} \times \hat{\phi} = \hat{\theta}$~~ $\hat{r} \times \hat{\theta} = \hat{\phi}$



Beware of definition of $\phi \longleftrightarrow \theta$ is often reversed.

What do the unit vectors mean?

In Cartesian coordinates,

$$\hat{x} = (1,0,0) \qquad \hat{y} = (0,1,0) \qquad \hat{z} = (0,0,1)$$

$$\Rightarrow \quad \nabla \cdot \hat{x} = \left(\frac{\partial 1}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial 0}{\partial z} \right)$$

$$= 0$$

$\Rightarrow$ Cartesian unit vectors are constant.

What about $\hat{\rho}$, $\hat{r}$, $\hat{\phi}$, and $\hat{\theta}$?

$$\vec{\rho} = (x, y, 0) \qquad \rho = \sqrt{x^2+y^2}$$

$$\hat{\rho} = \left(\frac{x}{\rho}, \frac{y}{\rho}, 0 \right)$$

$$= \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 0 \right)$$

$$= (\cos\phi, \sin\phi, 0)$$

$\hat{\phi}$ is $\perp$ to $\hat{\rho}$ and to $\hat{z}$

$$\hat{\phi} \cdot \hat{z} = 0 \qquad \hat{\phi} \cdot \hat{\rho} = 0$$

$$\hat{\phi} = \pm \left(\frac{-y}{\rho}, \frac{x}{\rho}, 0 \right)$$

<u>Choose sign for right handed</u>

If $\hat{\rho} = \hat{x}$, $\hat{\theta} = \hat{y}$

$$\hat{\rho} = \hat{x} \Rightarrow x = 1,\ y = 0$$

$$\hat{\phi} = \pm (0, 1, 0)$$ select +

$$\hat{\phi} = \left(\frac{-y}{\rho}, \frac{x}{\rho}, 0 \right) = -(\sin\phi, \cos\phi, 0)$$

What is $\nabla \cdot \hat{\phi}$ or $\nabla \hat{\phi}$? (zero??)

No, $\nabla \hat{\phi} = \left(-\nabla\left(\frac{y}{\rho}\right), \nabla\left(\frac{x}{\rho}\right), 0 \right) \neq 0$

The expressions ∇f, $\nabla \cdot \vec{A}$, $\nabla \times \vec{A}$ are scalars (numbers) or vectors (number + direction) that represent physical quantities. The are real things that are not affected by how they are represented mathematically.

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

$$= A_\rho \hat{\rho} + A_\phi \hat{\phi} + A_z \hat{z} \qquad \text{is still } \vec{A}$$

$$= A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi}$$

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= \frac{1}{\rho} \frac{\partial (\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$= \frac{1}{r^2} \frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta} \frac{\partial (A_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta} \frac{\partial A_\phi}{\partial \phi}$$

Note, to use these formulas, you have to completely express the function or field in the variables of the coordinate system.

$$x = \rho\cos\phi = r\sin\theta\cos\phi$$

$$y = \rho\sin\phi = r\sin\theta\sin\phi$$

$$z = z = r\cos\theta$$

Weirdly missing from front cover.

Consider, $\vec{E} = x\,\hat{\phi}$. Calculate $\nabla\cdot\vec{E}$

Method 1

$$\vec{E} = x\left(\frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}, 0\right)$$

$$= \left(\frac{-xy}{\sqrt{x^2+y^2}}, \frac{x^2}{\sqrt{x^2+y^2}}, 0\right)$$

$$\nabla\cdot\vec{E} = \frac{\partial}{\partial x}\left(\frac{-xy}{\sqrt{x^2+y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{x^2}{\sqrt{x^2+y^2}}\right)$$

$$= \frac{-y}{\sqrt{x^2+y^2}} + \frac{x^2 y}{(x^2+y^2)^{3/2}} - \frac{yx^2}{(x^2+y^2)^{3/2}} = \frac{-y}{\sqrt{x^2+y^2}}$$

Method 2

$$\vec{E} = \rho \cos\phi \, \hat{\phi}$$

$$= A_\rho \hat{\rho} + A_\phi \hat{\phi} + A_z \hat{z}$$

$$A_\rho = 0 \,, \quad A_\phi = \rho\cos\phi \,, \quad A_z = 0$$

$$\nabla \cdot \vec{E} = 0 + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + 0$$

$$= -\sin\phi = \frac{-y}{\sqrt{x^2+y^2}} \quad \checkmark$$

Changing Coordinate Systems - The exercise above showed one method of changing coordinate systems. Grab the representation of the unit vectors in the new frame and substitute. We can also exploit the fact that vectors are vectors.

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} = A_\rho \hat{\rho} + A_\phi \hat{\phi} + A_z \hat{z}$$

$$A_x = \hat{x}\cdot\vec{A} = A_\rho\,\hat{x}\cdot\hat{\rho} + A_\phi\,\hat{x}\cdot\hat{\phi} + A_z\,\hat{x}\cdot\hat{z}$$

$$= A_\rho \cos\phi - A_\phi \sin\phi$$