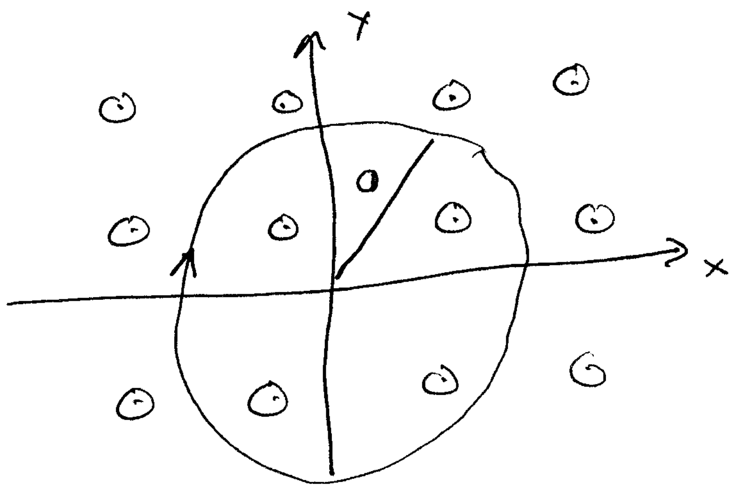
EMF Examples

Loop of radius a in x-y plane. Magnetic field $\vec{B} = \gamma t^2 \hat{z}$. Loop has resistance R. Compute current and back magnetic field at center.



(Sorry doing this after game, pretend its a circle)

$N = 1$

$$\text{Flux} = N\vec{B}\cdot\hat{n}\,A$$

$$= NBA = |B|\pi a^2$$

$$\Phi = \gamma \pi a^2 t^2$$

Flux Rule

$$\text{emf} = -\dot{\Phi} = -2\gamma\pi a^2 t$$

Lenz' Law - Discard the sign and use Lenz

(1) Flux out of page increasing.

(2) Induced current produces flux into page.

(3) By RHR, ~~clock~~ current clockwise

The current is $I = emf/R$ by Ohm's law.

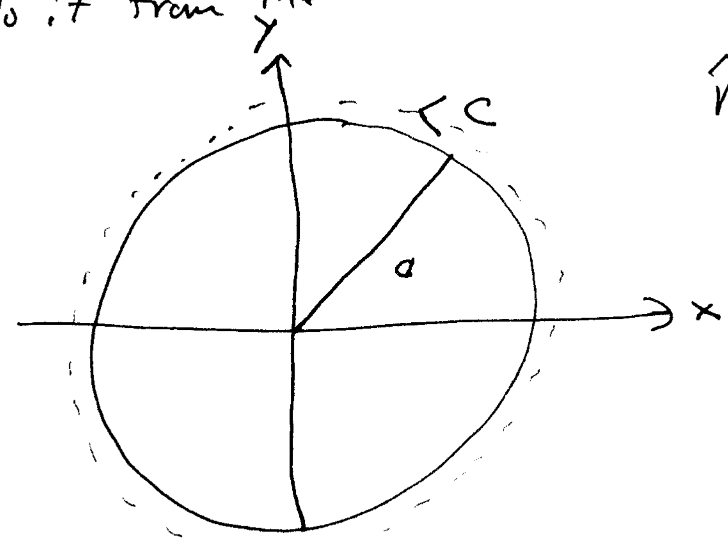
$$I = \frac{2\gamma\pi a^2 t}{R}$$

The magnetic field due to this current is given by the field of a circular loop

$$\vec{B}_{back} = -\frac{\mu_0 I}{2a}\hat{z} = \frac{-2\gamma\pi a^2 t\mu_0}{2aR}\hat{z}$$

$$= -\frac{\gamma\pi a t\mu_0}{R}\hat{z}$$

Now, let's do it from the definitions.



$\hat{n}$ upward

Let $\hat{n} = \hat{z}$ be the positive surface normal. By the right hand convention, ~~the emf will be positive if evalu~~ the curve C will be integrated in the counterclockwise direction.

The flux is

$$\Phi = \int_S \vec{B} \cdot d\vec{a} = \int_S \vec{B} \cdot \hat{n}\, da$$

where $d\vec{a} = \hat{n}\, da$ which points in the $\hat{z}$ direction toward the positive normal.

$$\Phi = \int_S \gamma t^2 \hat{z} \cdot \hat{z}\, da = \gamma t^2 \int_S da = \gamma t^2 \pi a^2$$

$$\text{The emf} = -\dot{\Phi} = -\frac{d}{dt} \gamma t^2 \pi a^2 = -2\gamma t \pi a^2$$

The emf is defined as

$$\text{emf} = \oint_C \vec{E} \cdot d\vec{l} = -2\gamma t \pi a^2$$

Since C is positive in the counterclockwise direction, and emf is negative, $\vec{E}$ points in clockwise direction.

The current flows in the direction of $\vec{E}$.

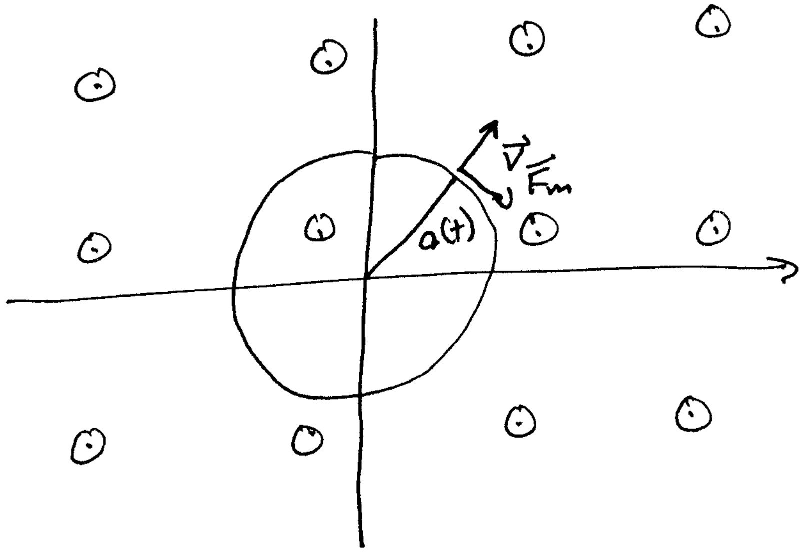
Ex Take the previous example an rotate the loop by 30° about the y-axis. What happens?

$$\Phi_m = N\vec{B}\cdot\hat{n}\,A$$

$$= BA\cos\theta = BA\cos 30$$

The emf is reduced by $\cos 30° = \sqrt{3}/2$

Ex Consider the same loop, but now let the magnetic field be constant and let the loop grow as $a(t) = a_0 + vt$. $\vec{B} = B_0\hat{z}$



$$\Phi = N\vec{B}\cdot\hat{n}\,A = B_0\pi a(t)^2$$

$$= B_0\pi(a_0+vt)^2$$

Flux Rule

$$\text{emf} = -\dot{\Phi}_m = 2\pi B_0 v (a_0 + vt)$$

Lenz Law - Flux out of the page is increasing again, so clockwise current

We should be ~~the~~ able to compute the emf directly. The emf is the work per unit charge ~~to move a~~ done by the fields as a positive charge q moves around C.

$$\frac{W}{q} = \text{emf} = \frac{2\pi a(t) F_m}{q}$$

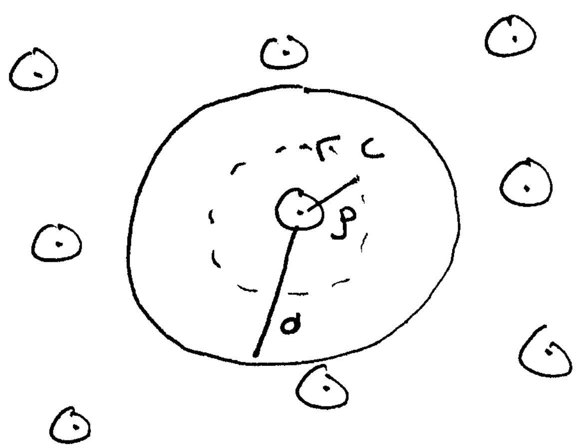
$$|\vec{F}_m| = q\, \vec{v} \times \vec{B} = q v B_0 \qquad \text{clockwise}$$

$$\text{emf} = 2\pi\, a(t)\, v B_0$$

$$= 2\pi v B_0 (a_0 + vt)$$

Why didn't we check the previous example with direct calculation? In that example, the emf was provided by the electric field which we didn't know.

Let's check it anyway



$\vec{B} = \gamma t^2 \hat{z}$

counter clockwise positive

Let region be cylindrical, $\vec{E}$ fields circular

$$\oint \vec{E} \cdot d\vec{l} = 2\pi\rho\, E(\rho) = -\dot{\Phi}_m$$

$$= -\frac{d}{dt}(NBA)$$

$$= -\frac{d}{dt}\left(\gamma t^2 \pi a^2\right) = -2\gamma t \pi a^2$$

clockwise

$$E(\rho) = \frac{-2\gamma t \pi a^2}{2\pi\rho} = -\frac{\gamma t a^2}{\rho}$$

~~check with~~

Check with Lenz Law

Flux out of page increasing

Induced current, which flows in direction of $\vec{E}$, creates flux that opposes change

$\Rightarrow$ clockwise current

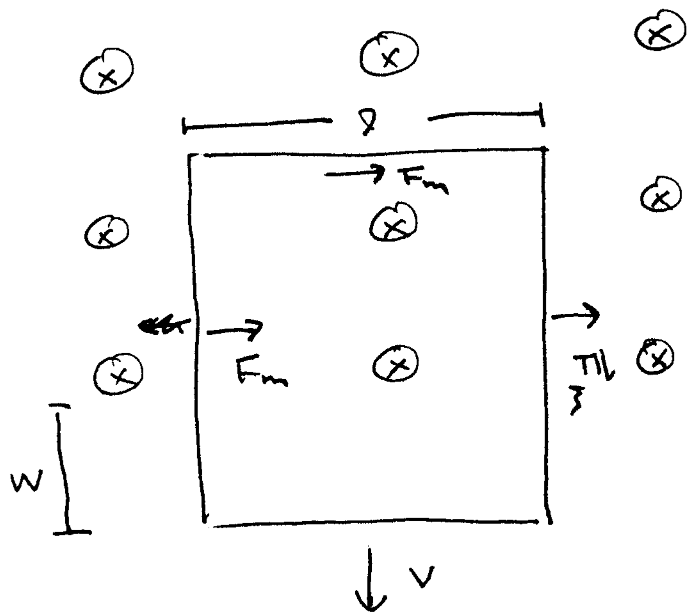
The emf is the work per charge around the loop

$$\text{emf} = \frac{W}{q} = \oint \vec{E}(a) \cdot d\vec{\ell}$$

$$= (2\pi a)\left(\frac{-\gamma t a^2}{a}\right)$$

$$= -\frac{2\pi\gamma t}{\cancel{\ }} a^2$$

Ex A deceptive problem



$N = 200$

$B = \frac{1}{4}\,T$

$R = 1\,\Omega$

Method 1 Flux Rule

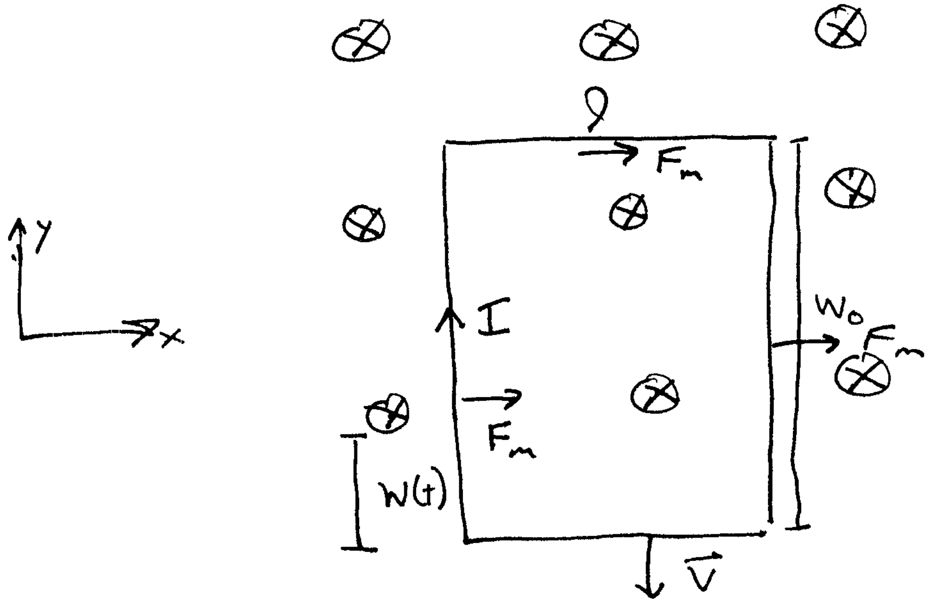
$$\Phi_m = N B A(t) = N B \ell w(t)$$

$$\text{emf} = -\dot{\Phi}_m = -N B \ell \frac{dw}{dt} = -N B \ell v$$

Method 2 Motional EMF

$$\text{emf} = N \oint (\vec{v} \times \vec{B} \cdot d\vec{\ell}) = \qquad N v B \ell$$

Ex A deceptive problem. Pull rectangular loop out of a magnetic field.



N turns

$\vec{B} = B_o \hat{z}$

$w(t) < W_o$

Method I Flux Rule

$$\Phi_m = N l \,(W_o - W(t))\, B$$

$$\dot{\Phi}_m = -N l B \frac{dW}{dt} = -N l B v$$

$$\text{emf} = -\dot{\Phi}_m = N l B v$$

Direction - Flux is decreasing, so current flows to produce flux to oppose this decrease (into page). This gives clockwise current by RHR.

Direction 2 - For Φ to be positive for given $\vec{B}$, clockwise represents positive direction. emf > 0, so current flows clockwise.

Method II   Motional emf

$\vec{F}_m$ only points along wire at top

$$\text{emf} = N \oint (\vec{v} \times \vec{B}) \cdot d\vec{l} = N v B l \quad \text{clockwise.}$$

Change problem slightly — Let $\vec{B} = B_0 t \, \hat{z}$

Method I   Flux rule

$$\Phi_m = N l (w_0 - w(t)) B_0 t$$

$$\dot{\Phi}_m = N l (w_0 - w(t)) B_0 - N l B_0 t v$$

Let $w = vt$

$$\dot{\Phi}_m = N l (w_0 - vt) B_0 - N l B_0 v t$$

$$= N l w_0 B_0 - 2 N l B_0 v t$$

$$\text{emf} = -\dot{\Phi}_m = -N l B_0 (w_0 - 2vt)$$

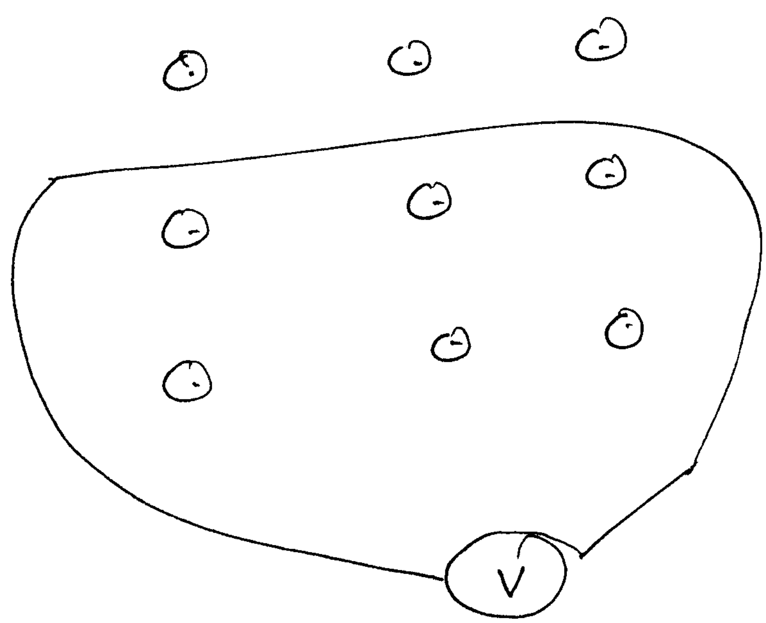
$vt < w_0$ current still clockwise.

Method II - Motional emf

$$\cancel{\mathcal{E}}\ \text{emf} = N v B l = N v B_0 t l$$

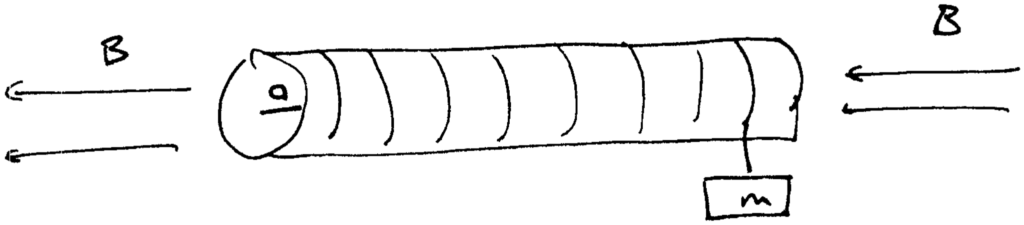
Methods not equal, motional emf does not account for electric field.

Now consider,



If B = constant we solved it. If $B \neq$ constant it is unsolvable.

GRE Question Consider coil wound on rotating cylinder.



The mass exerts a constant torque $\tau = mga$ on the cylinder. The moment of inertia of a cylinder is, $I = Ma^2$ if the cylinder is a thin shell.

For a constant torque,

$$\tau = I\dot{\omega}$$

$$\omega(t) = \omega_0 + \frac{\tau}{I} t$$

$$N = \cancel{2\pi a} \quad \frac{L_{wire}}{2\pi a}$$

$L_{wire} \equiv$ Total Length of wire

$$\frac{dN}{dt} = \frac{1}{2\pi a} \frac{dL_{wire}}{dt}$$

$$\frac{dL_{wire}}{dt} = a\omega = a\left(\omega_0 + \frac{\tau}{I} t\right)$$

Let $\omega_0 = 0$

$$\frac{dN}{dt} = \frac{1}{2\pi a}\frac{dL_{wire}}{dt} = \frac{a\omega}{2\pi a} = f$$

$$= \frac{\tau}{2\pi I}t = \frac{mga}{2\pi M a^2}t$$

$$= \left(\frac{g}{2\pi a}\right)\left(\frac{m}{M}\right)t$$

$$\Phi = N B A = N(t)\, B \pi a^2$$

$$\text{emf} = -\dot{\Phi} = B\pi a^2 \frac{dN}{dt} = \left(B\pi a^2\right)\left(\frac{g}{2\pi a}\right)\left(\frac{m}{M}\right)t$$

$$= \left(\frac{Bga}{2}\right)\left(\frac{m}{M}\right)t$$

$$[\text{emf}] = \frac{T m^2}{s} = \frac{Wb}{s} \checkmark$$

Actual Gre Question $\qquad \omega = \text{constant}$

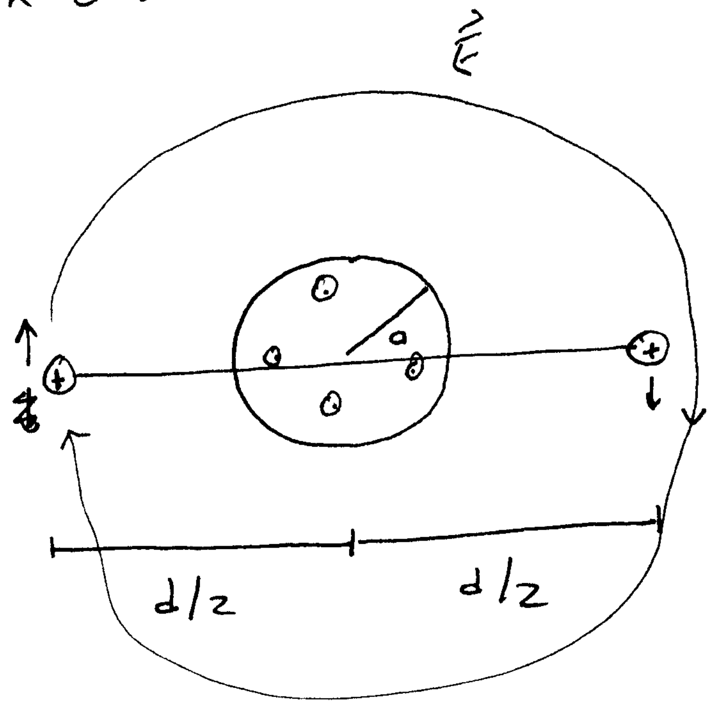
$$\frac{dN}{dt} = \frac{\omega}{2\pi} = f$$

$$\text{emf} = -\dot{\Phi} = -\frac{dN}{dt} BA = -fBA$$

$$= \frac{B\pi a^2 \omega}{2\pi} = \frac{Ba^2\omega}{2}$$

Crazy GRE Question

Let region $\rho < a$ contain magnetic field $\vec{B} = B_0 \hat{z}$ the goes from B_0 to 0 in time Δt. Compute the change in angular momentum of a charge ~~dipole $\pm q$~~ stick of length $d > 2a$



$$\tau = \frac{dL}{dt}$$ where τ torque and L angular momentum.

$$\Delta L = \int_0^{\Delta t} \tau \, dt$$

Is force on charges non-zero.

$$\tau = 2\left(\frac{d}{2} F_e\right) = d\,q\,E(d/2)$$

~~A~~As before,

$$2\pi p\, E(p) = -\dot{\Phi}_m$$

$$\Phi_m = NBA = B\pi a^2$$

$$= B(t)\,\pi a^2$$

$$\dot{\Phi}_m = \pi a^2 \frac{dB}{dt}$$

$$E(p) = \frac{a^2}{2p}\frac{dB}{dt} \qquad \text{sign discarded}$$

$$E(d/2) = \frac{a^2}{d}\frac{dB}{dt}$$

$$\tau = d\,q\,E(d/2) = q a^2 \frac{dB}{dt}$$

$$\Delta L = \int_0^{\Delta t} \tau\, dt = q a^2 \int \frac{dB}{dt}\, dt$$

$$= q a^2 \int_{B_0}^{0} dB = -q a^2 B_0$$