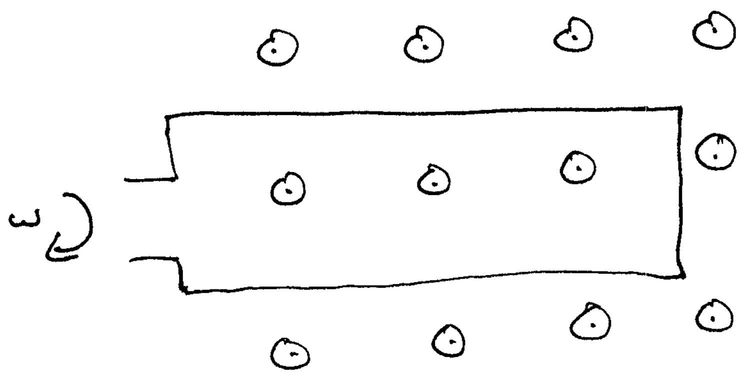
Finishing Faraday

Normal Generators – Coil spinning in uniform magnetic field



Let coil start with angle θ_0 between $\vec{B}$ and $\hat{n}$ and spin at constant rate ω.

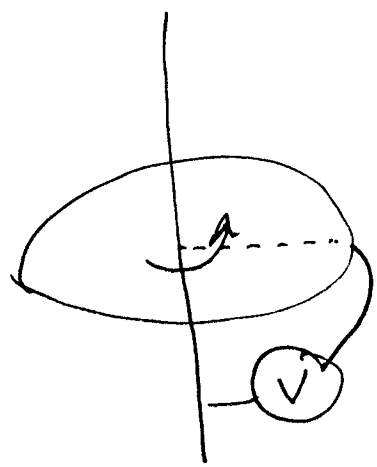
$$\Phi = N\vec{B}\cdot\hat{n}\,A = NBA\cos\theta = NBA\cos(\omega t + \theta_0)$$

Flux Rule

$$\text{emf} = -\dot{\Phi} = -NBA\omega\sin(\omega t + \theta_0)$$

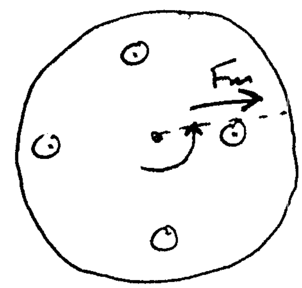
Output Peak emf, $NBA\omega$, faster spin, more volts.

Faraday's Dynamo



Magnetic field upward.

Top View



Radius a

Compute motional emf along dashed line.

$$v = \omega r$$

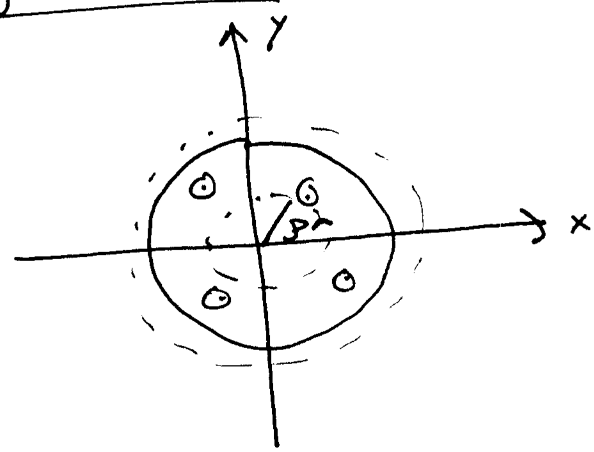
$$\text{emf} = \oint \vec{v} \times \vec{B} \cdot d\vec{l} = \int_0^a vB\,dr$$

$$= \omega B \int r\,dr$$

$$= \frac{\omega B r^2}{2}$$

$\underline{Ex}$ Magnetic field created by uniform current density $\vec{J} = \gamma \rho t \hat{z}$ in cylindrical region $\rho < a$. Compute emf around closed path of length l in wire under assumption $(E_z(0) = 0)$.

Compute Magnetic Field



$\underline{\rho < a}$ Use Amperian path of radius ρ. Since the system and current is cylindrical, the field will be circular.

Ampere's Law $(I_d = 0\ ?)$

$$\oint \vec{B} \cdot d\vec{l} = 2\pi\rho\, B_\phi(\rho) = \mu_0 I_{enc} = \mu_0 \int_0^\rho \vec{J} \cdot da$$

$$I_{enc} = \int_0^\rho \vec{J} \cdot d\vec{a} = \int_0^\rho d\rho \int_0^{2\pi} \rho\, d\phi\, \gamma \rho t$$

$\hat{n} = \hat{z}$

$\vec{J} \cdot d\vec{a} = \gamma \rho t\, da$

$da = dr\, r\, d\phi$

$$I_{enc} = 2\pi\gamma t \int_0^{\rho} \rho^2 d\rho = \frac{2\pi\gamma\rho^3}{3}$$

Ampere's Law

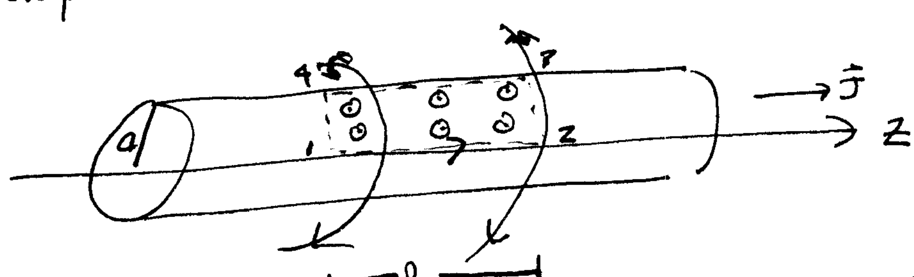
$$2\pi\rho\, B_\phi(\rho) = \mu_0 I_{enc} = \frac{2\pi\mu_0\gamma\rho^3 t}{3}$$

$$B_\phi(\rho) = \frac{\mu_0\gamma\rho^2 t}{3}$$

$$\vec{B}_\phi(\rho) = \frac{\mu_0\gamma\rho^2 t}{3}\hat{\phi}$$

We have handled the signs of everything carefully so this should jive with the RHR which indicates that current should flow counterclockwise. The $\hat{\phi}$ direction is counter clockwise.

Now compute the emf and the electric field.



The flux out of the page is increasing, so the field would drive an induced current whose field would oppose the change in flux. Therefore the field is generally clockwise around path.

No net charge is given is the problem, so we may assume $\Gamma = 0$ at $t=0$. Let Γ be the volume charge density to avoid confusion with Γ. We need to check whether the current produces a net charge.

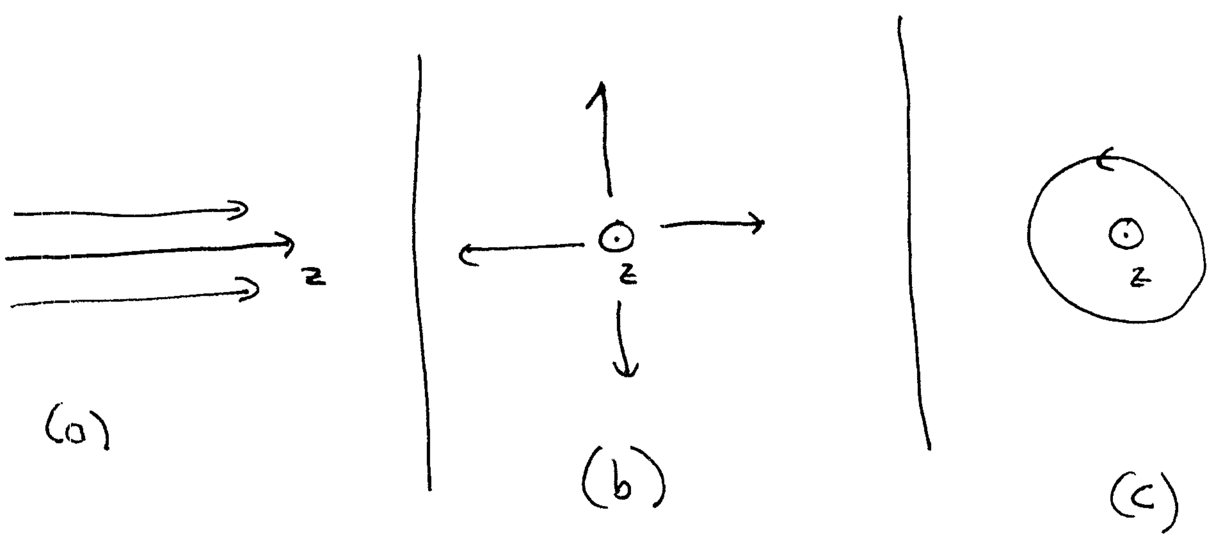
<u>Continuity Eqn</u>

$$\nabla \cdot \vec{J} + \frac{\partial \Gamma}{\partial t} = 0$$

$$\nabla \cdot \vec{J} = 0 \quad \Rightarrow \quad \Gamma = \text{constant} = 0$$

No net charge.

$\Rightarrow$ $\nabla \cdot \vec{E} = 0$ and the electric field must be divergence free. Consider our possible field shapes by symmetry



In case (b), $\nabla \cdot \vec{E} \neq 0$.

In case (c), $\oint \vec{E} \cdot d\vec{l} \neq 0 \Rightarrow \dot{\Phi}_m \neq 0$

but this is not true because the magnetic field is also circles.

$\Rightarrow$ The field has shape (a).

Do the integral,

$$\oint \vec{E} \cdot d\vec{l} = \underbrace{\int_{1\to 2}}_{\text{zero by assumption}} + \int_{2\to 3} + \int_{3\to 4} + \int_{4\to 1}$$

0 $\quad \vec{E} \perp d\vec{l}$

$$\oint \vec{E} \cdot d\vec{l} = - E_z(\rho)\, l$$

$$= - \dot{\Phi}_m \quad \text{by the Flux rule.}$$

<u>Compute flux through surface</u> $\qquad da = l\, d\rho$

$$\Phi_m = \int_S \vec{B} \cdot d\vec{a} = l \int_0^{\rho} d\rho \, \frac{\mu_0 \gamma \rho^2 t}{3}$$

$$\Phi_m = \frac{\mu_0 l \gamma t \rho^3}{9}$$

Flux rule

$$\text{emf} = \oint \vec{E}\cdot d\vec{l} = -E_z(\rho)\, l$$

$$= -\dot{\Phi}_m = -\frac{\mu_0 l \gamma \cancel{t} \rho^3}{9}$$

$$\vec{E}_z(\rho) = \frac{\mu_0 \gamma \cancel{t} \rho^3}{9}\,\hat{z}$$

Note $I_d = 0$.

Computed under assumptions + guesses

$$\vec{J} = \gamma \rho t\, \hat{z}$$

$\Pi = 0$

$$\vec{B} = \frac{\mu_0 \gamma \rho^2 t}{3}\,\hat{\phi}$$

$$\vec{E} = \frac{\mu_0 \gamma \rho^3}{9}\,\hat{z}$$