Maxwell's Equations

Gauss $\quad \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \longleftrightarrow \oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$

No Mag Monopoles $\quad \nabla \cdot \vec{B} = 0 \longleftrightarrow \oint_S \vec{B} \cdot d\vec{a} = 0$

Faraday's Law $\quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \longleftrightarrow \oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{a}$

Amperes Law $\quad \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 (I_{enc} + I_d)$$

$$I_d = \varepsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot d\vec{a}$$

Force Law (Lorentz)

$$\vec{F} = q\vec{E} + q\,\vec{v} \times \vec{B}$$

Static cases

$\frac{\partial \rho}{\partial t} = 0 \qquad \vec{E} = \sum \frac{Q_i}{4\pi\varepsilon_0 r^2} \hat{r}$

Electrostatic

$\vec{B} = \int d\vec{B} = \int \frac{\mu_0}{4\pi} \frac{\vec{J} \times \hat{r}}{r^2}$

Magnetostatic $\quad \nabla \cdot \vec{J} = 0$

Maxwell's equations and the Lorentz force are true at all points in space. We will make additions to them as the course goes on, but these aid calculations and do not add physics.

Let's play with them a bit, before we test our fields.

Try taking div and curl of each equation.

$$\nabla \cdot (\nabla \times \vec{E}) = -\frac{\partial}{\partial t} \nabla \cdot \vec{B} = 0$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \text{ for all vectors.}$$

This is nothing new, since $\nabla \cdot \vec{B} \cdot 0$.

$$\nabla \cdot (\nabla \times \vec{B}) = \mu_0 \nabla \cdot \vec{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} = 0$$

Sub Gauss

$$\nabla \cdot \vec{J} + \varepsilon_0 \frac{\partial}{\partial t} \frac{\rho}{\varepsilon_0} = 0$$

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Continuity Eqn $\longrightarrow$ Charge is conserved.

Now try curls

$$\nabla \times (\nabla \times \vec{E}) = -\frac{\partial}{\partial t} \nabla \times \vec{B}$$ curl of Faraday

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} (\nabla \times \vec{B})$$ vector identity

$$\nabla\left(\frac{\rho}{\varepsilon_0}\right) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} (\nabla \times \vec{B})$$ Gauss'

$$= -\mu_0 \frac{\partial \vec{J}}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$ Ampere

Re-arrange

$$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial \vec{J}}{\partial t} + \nabla\left(\frac{\rho}{\varepsilon_0}\right)$$

Wave equation with wave velocity

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \times 10^8 \, \mathrm{m/s}$$

Curl of Ampere's Law

$$\nabla \times (\nabla \times \vec{B}) = \mu_0 \nabla \times \vec{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \times \vec{E})$$

$$\nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = \mu_0 \nabla \times \vec{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \times \vec{E})$$

sub Faraday

$$-\nabla^2 \vec{B} = \mu_0 \nabla \times \vec{J} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

$$-\mu_0 \nabla \times \vec{J} = \nabla^2 \vec{B} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

Another wave equation with speed $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

Let's try our fields to see if the can be real fields.

$$\vec{J} = \gamma \rho t \, \hat{z} \qquad\qquad 0 = \Gamma$$

$$\vec{B} = \frac{\mu_0 \gamma \rho^2 t}{3} \hat{\phi}$$

$$\vec{E} = \frac{\mu_0 \gamma \rho^3}{9} \hat{z}$$

To be an electromagnetic field, the fields must satisfy each Maxwell equation.

<u>Gauss</u> $\quad \nabla \cdot \vec{E} = \dfrac{\Gamma}{\varepsilon_0}$

$\nabla \cdot \vec{E} = 0$ by observation

<u>No Monopoles</u> $\quad \nabla \cdot \vec{B} = 0$ by observation

<u>Faraday</u>

$$\nabla \times \vec{E} = -\frac{\partial E_z}{\partial \rho} \hat{\phi} = \frac{-\mu_0 \gamma \rho^2}{3} \hat{\phi}$$

$$= -\frac{\partial \vec{B}}{\partial t} = -\frac{\mu_0 \gamma \rho^2}{3} \hat{\phi} \quad \checkmark$$

Ampere

$$\nabla \times \vec{B} = \frac{1}{\rho}\left(\frac{\partial(\rho B_\phi)}{\partial \rho}\right)\hat{z}$$

$$= \frac{1}{\rho}\,\frac{\mu_0 \gamma \cancel{\rho} t}{3}\,\frac{\partial}{\partial \rho}\rho^3\,\hat{z}$$

$$= \frac{1}{\rho}\,\mu_0 \gamma t\,\rho^2\,\hat{z} = \mu_0 \gamma t \rho\,\hat{z}$$

$$= \underbrace{\mu_0 \vec{J}}_{\mu_0 \gamma t \rho \hat{z}} + \underbrace{\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}_{0}$$

✓

These could be fields and the current looks like something we could establish.

Monopoles

Maxwell's equations would be more symmetric if their existed a magnetic charge ρ_m and corresponding magnetic current $\vec{J}_m$.

$$\nabla \cdot \vec{E} = \rho/\epsilon_0$$

$$\nabla \cdot \vec{B} = \rho_m$$

$$\nabla \times \vec{E} = -\vec{J}_m - \frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Such equation would predict a conserved magnetic charge

$$\nabla \cdot (\nabla \times \vec{E}) = 0 = -\nabla \cdot \vec{J}_m - \frac{\partial \nabla \cdot \vec{B}}{\partial t}$$

$$= -\nabla \cdot \vec{J}_m - \frac{\partial \rho_m}{\partial t} = 0$$

Continuity Eqn for Magnetic Charge

$$\nabla \cdot \vec{J}_m + \frac{\partial \rho_m}{\partial t} = 0$$

Modern Physics ▸ Particle Physics ▸ Particles

Magnetic Monopole

A hypothetical particle which would be the magnetic analog of charged particles carrying magnetic "charge." Dirac (1931) showed that the existence of magnetic monopoles could explain the quantization of electric charge, and showed that if they exist, they must carry a "magnetic charge" that is an integer multiple of $g_D = 68.5e$ in cgs units (Jeon and Longon 1995).

Magnetic monopoles have never been observed experimentally, and their absence results in the Maxwell equation

$$\nabla \cdot \mathbf{B} = 0.$$

Some grand unified theories (GUTs) predict the existence of small numbers of these particles (t'Hooft 1974, Polyakov 1974). The charge on magnetic monopoles predicted by GUTs is either 1 or $2g_D$ (Jeon and Longo 1995). The best experimental upper limit, obtained by searching for induced currents in superconducting wires, is 1 monopole per 10^{29} nucleons (Jeon and Longo 1995). The upper limit on the monopole mass is 10^{26} eV, or 0.2 μg.

SEE ALSO: Magnetic Dipole

REFERENCES:

Dirac, P. A. M. *Proc. Roy. Soc. London A* **133**, 60, 1931.

Jeon, H. and Longo, M. J. "Search for Magnetic Monopoles Trapped in Matter." *Phys. Rev. Lett.* **75**, 1443-1446, 1995.

Polyakov, A. [Russian]. *Pis'ma Zh. Eksp. Teor. Fiz.* **20**, 430, 1974. Reprinted in *JETP Lett.* **20**, 194, 1974.

t'Hooft, G. *Nucl. Phys.* **B79**, 276, 1974.

Section 5.2 uses a rather flaky derivation of the monopole charge as

$$Q_m = \frac{h}{e} = 4.1 \times 10^{-15}\ Tm^2$$

and the predicted monopole angular momentum with an electron as

$$\Delta L = \frac{2e\, Q_m}{4\pi} = \frac{\hbar}{2}$$

for EVERY electron in the universe.

Which is fine because the current limit on the number of monopoles in the universe is less than

1 per 10^{29} nucleons