Energy

We have been working with potential as a calculation aid. They have a more natural interpretation in terms of the energy of a system of charges.

If $\frac{\partial \vec{A}}{\partial t} \rightarrow \frac{\partial \vec{B}}{\partial t} = 0$,

$$\vec{E} = -\nabla V \implies \nabla \times \vec{E} = 0$$

$\implies$ The electric field is conservative

$\implies$ Work done to move a particle around a closed path is zero.

$$W \equiv \oint_c \vec{F} \cdot d\vec{\ell} \qquad \vec{F} = -q\vec{E}$$

$$= -q \int \vec{E} \cdot d\vec{\ell}$$

$$= q \int_c \nabla V \cdot d\vec{\ell} = q \Delta V = q\left(V(\vec{r}_2) - V(\vec{r}_1)\right)$$

$\Rightarrow$ The potential difference ΔV is the work per unit charge to move a charge along a path from $\vec{r}_1$ to $\vec{r}_2$, $\Delta V(\vec{r}_1, \vec{r}_2) = V(\vec{r}_2) - V(\vec{r}_1)$

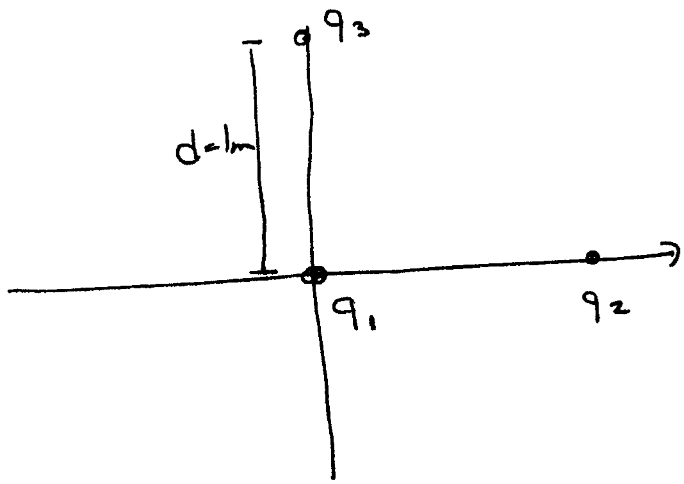
$\Rightarrow$ Potential is the potential difference with respect to some reference point $\vec{r}_0$.

$$V(\vec{r}) = \Delta V(\vec{r}, \vec{r}_0)$$

The energy of a system is the work to assemble it.

$$U = \sum W_i$$

Ex Energy of 3 $q = 1nC$ charges placed as below



Energy is the work to build the system

$$U = W_1 + W_2 + W_2$$

$$W_1 = 0$$

$$W_2 = q_2 \Delta V_1 (\vec{r}_2, \vec{r}_0)$$

$$= q_2 \left(\frac{q_1 k}{d_{12}} \right) = \frac{k q_1 q_2}{d_{12}}$$

Potential of Point Charge

$$V = \frac{kq}{r} \qquad (V(\infty) = 0)$$

$$W_3 = q_3 \Delta V_1 (\vec{r}_3, \vec{r}_0) + q_3 \Delta V_2 (\vec{r}_3, \vec{r}_0)$$

$$= q_3 \left(\frac{k q_1}{d_{13}} \right) + q_3 \left(\frac{k q_2}{d_{23}} \right)$$

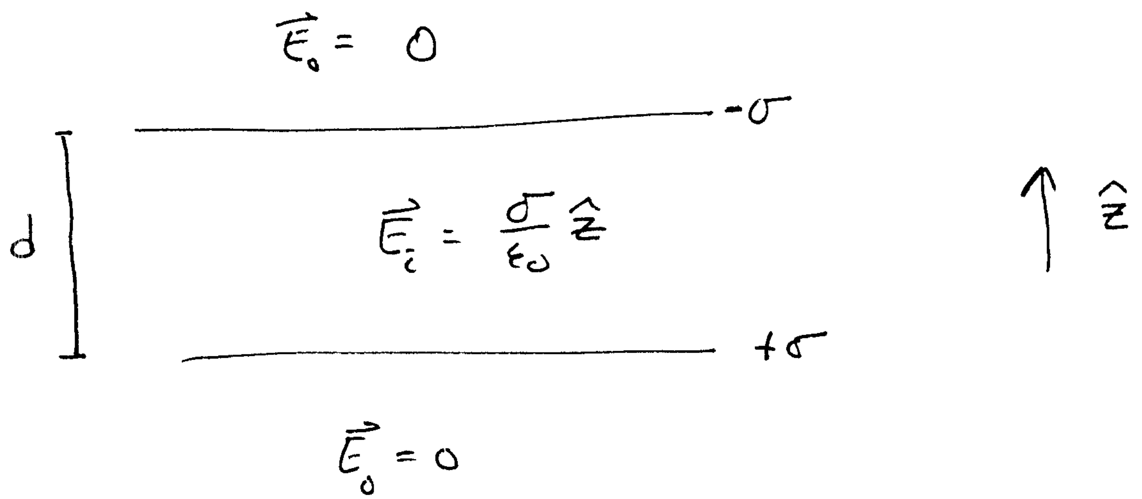
$$U = \frac{k q_1 q_2}{d_{12}} + \frac{k q_1 q_3}{d_{13}} + \frac{k q_2 q_3}{d_{23}}$$

$$d_{12} = d_{13} = 1\,\text{m} \qquad d_{23} = \sqrt{2}\,\text{m}$$

Note, distances are positive.

<u>Ex</u> Work per unit area to construct, equal and opposite planes of charge, σ.

The final fields are



We can construct the system by taking the two planes when they are touching and separating them.

If the planes have area A, the force the bottom plane exerts on the top plane is

$$\vec{F}_t = Q\left(\frac{\vec{E}_i + \vec{E}_o}{2}\right)$$

where we have to average the fields because of the discontinuity.

The work done is $W = F_t d$

$$W = dQ\frac{E_i}{2} = d\sigma A\frac{E_i}{2}$$

$$= (dA)(\varepsilon_0 E_i)\frac{E_i}{2}$$

$$= (dA)\frac{1}{2}\varepsilon_0 E_i^2 = \text{Total Energy}$$

This energy resides in the fields between the plates, so the energy density of the field is

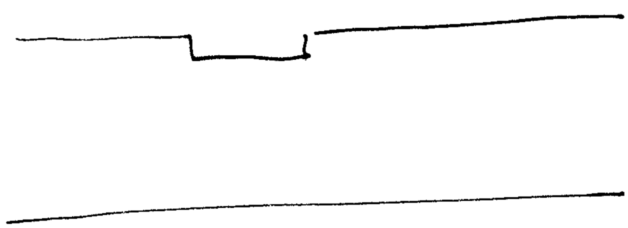
$$\frac{W}{V} = \frac{1}{2}\varepsilon_0 E^2$$

which must be ~~the~~ the energy density of any field.

If a small modification, δx, of a system leads to a change in energy δU, there will be a force in the direction of the modification

$$F = -\frac{\delta U}{\delta x}$$

Consider a small deformation of the top plate of depth $-\delta z$ and area A



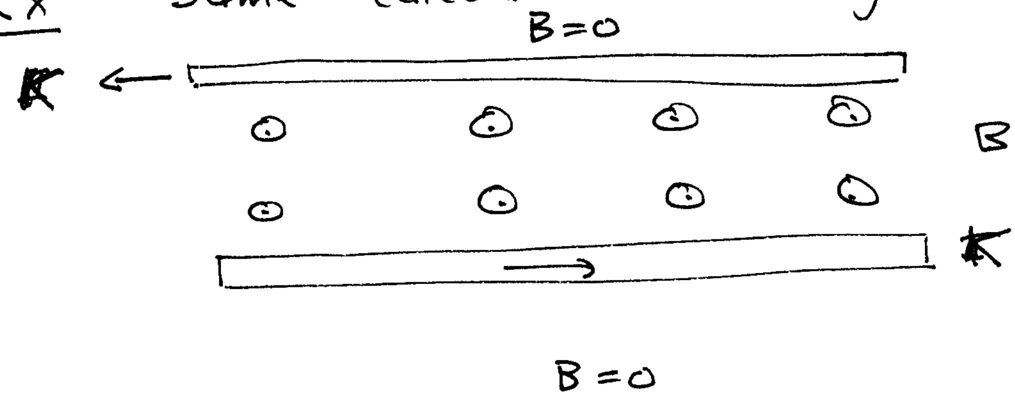
The change is energy is $\delta U = -\delta z A\left(\frac{1}{2}\varepsilon_0 E^2\right)$

so the force is $F = -\frac{\delta U}{\delta z} = A\left(\frac{1}{2}\varepsilon_0 E^2\right)$ downwary

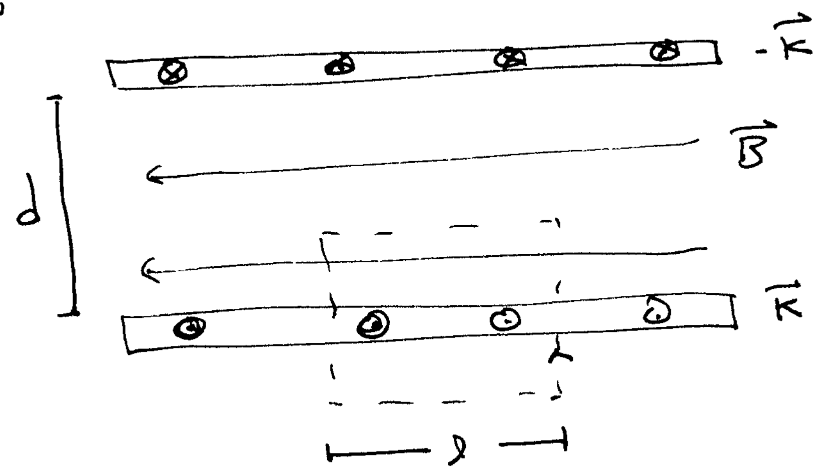
or the pressure is $P = \frac{F}{A} = \frac{1}{2}\varepsilon_0 E^2$

$\Rightarrow$ The energy density of the fields is the pressure at the surface.

Ex Same calculation with magnetic fields.



End View



$$\oint \vec{B}\cdot d\vec{l} = B_i\, l = \mu_0 l K$$

$$B_i = \mu_0 K$$

The force the bottom plane exerts on the top plane for an area $A = (L \times L)$

For a strip of current dI of Length L

$$d\vec{F} = dI\,\vec{L} \times \left(\frac{\vec{B}_i + \vec{B}_o}{2}\right)$$

Everything is at right angles,

$$dF = dI\, L \frac{B_i}{2} \qquad dI = K\, dy$$

$$F = K L^2 \frac{B_i}{2}$$

$$W = F d = (L^2 d)\, K \frac{B_i}{2}$$

$$= (L^2 d) \frac{B_i^2}{2\mu_0}$$

Energy Density in the magnetic field is

$$\frac{dU}{dV} = \frac{B^2}{2\mu_0}$$

which will also be the magnetic pressure.

$\Rightarrow$ The energy densities do not require the assumptions of a static field.

Let's re-express the energy in terms of the potentials.

$$W = \frac{\varepsilon_0}{2} \int E^2 \, dv$$

$$= -\frac{\varepsilon_0}{2} \int \nabla V \cdot \vec{E} \, dv \qquad \text{for static fields.}$$

$$= \underbrace{+\frac{\varepsilon_0}{2} \int V \, \nabla \cdot \vec{E} \, dv}_{\text{Gauss}} - \underbrace{\frac{\varepsilon_0}{2} \int \nabla \cdot (V E) \, dv}_{\text{Divergence Thm}}$$

$$\frac{1}{2} \int \rho V \, dv \quad - \frac{\varepsilon_0}{2} \int_S V E \cdot d\vec{a}$$

If field on potential well behaved at ∞, the last term is zero. $\Rightarrow V(\infty) = 0$.

$$W = \frac{1}{2} \int \rho V \, dv$$

$$\text{or} \quad \frac{dW}{dv} = \frac{\rho V}{2} \qquad \text{energy density}$$

If all the charge is at the same potential $W = \frac{QV}{2}$

Let's do the same thing for the magnetic energy.

$$W = \frac{1}{2\mu_0} \int \vec{B} \cdot \vec{B} \, dv$$

$$= \frac{1}{2\mu_0} \int \vec{B} \cdot \nabla \times \vec{A} \, dv$$

$$= \underbrace{\frac{1}{2\mu_0} \int \vec{A} \cdot \nabla \times \vec{B} \, dv}_{\text{Ampere}} + \underbrace{\frac{1}{2\mu_0} \int \nabla \cdot (\vec{A} \times \vec{B}) \, dv}_{\text{Divergence}}$$

$$\frac{1}{2\mu_0} \int \vec{A} \cdot (\mu_0 \vec{J}) \, dv + \underbrace{\frac{1}{2\mu_0} \int_s \vec{A} \times \vec{B} \cdot d\vec{a}}_{\substack{0 \\ \text{if we behaved}}}$$

$$W = \frac{1}{2} \int \vec{A} \cdot \vec{J} \, dv \quad \text{if static}$$

Energy density ~~move~~ magnetic field

$$\frac{dW}{dv} = \frac{1}{2} \vec{A} \cdot \vec{J}$$

Ex (6-14) Calculate energy / length in a infinite solenoid.

Method I $\quad B = N' I \mu_0 = K \mu_0$

$(K = N' I)$

$$\text{Energy Density} = \frac{1}{2\mu_0} B^2$$

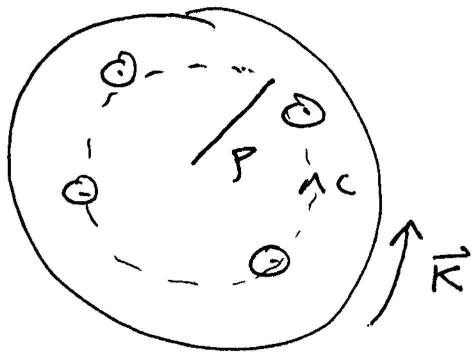
$$= \frac{1}{2\mu_0} (K \mu_0)^2 = \frac{1}{2} \mu_0 K^2$$

Note, this is also the outward pressure.

The energy per unit length,

$$\frac{U}{l} = A \left(\frac{1}{2} \mu_0 K^2 \right)$$

End View



Compute $\vec{A}$

Must be cylindrical by symmetry

$$\oint \vec{A} \cdot d\vec{l} = 2\pi\rho A = \Phi_m = \pi\rho^2 B$$

$$\vec{A} = \frac{\rho B}{2}\hat{\phi}$$

$$= \frac{\rho \mu_0 K}{2}\hat{\phi}$$

$$\vec{K} = K\hat{\phi}$$

The energy stored in a length l of the current is

$$W = \frac{1}{2}\int \vec{A} \cdot \vec{J}\, dv = \frac{1}{2}\int \vec{A} \cdot \vec{K}\, ds$$

Let the solenoid have radius a. At this radius, $\vec{A} = \frac{a \mu_0 K}{2} \hat{\phi}$

$$\vec{K} \cdot \vec{A} = \frac{a \mu_0 K^2}{2}$$

$$W = \frac{1}{2} \int \frac{a \mu_0 K^2}{2} \, da$$

$$= \frac{a \mu_0 K^2}{4} \int da$$

$$= \frac{a \mu_0 K^2}{4} \, 2 \pi a l$$

$$\frac{W}{l} = (\pi a^2) \frac{\mu_0 K^2}{2}$$