PHYS 3414 - Electricity and Magnetism- Final Exam 2

All problems are worth 25 points. Turn in solutions to four of the six problems to be graded. If you turn in more than four solutions, I will grade the first four. You are allowed to drop one-half of a test, so I will take the first two problems turned in as the first half-test and the second two problems turned in as the second half-test.

1 A wire of length 2ℓ runs along the x-axis and is centered at the origin. The wire is thinner in the middle than at the two ends. The cross-sectional area of the wire is given by $A(x) = A_0(a^2 + x^2)$, where A_0 and a are constants. Compute the resistance of the wire if it has conductivity σ.

2 A cylindrical conductor of radius a has a current density that increases with radius, $\vec{J}(\rho) = J_0\rho^2\hat{z}$. The current density is zero outside the wire. Compute the magnetic field everywhere.

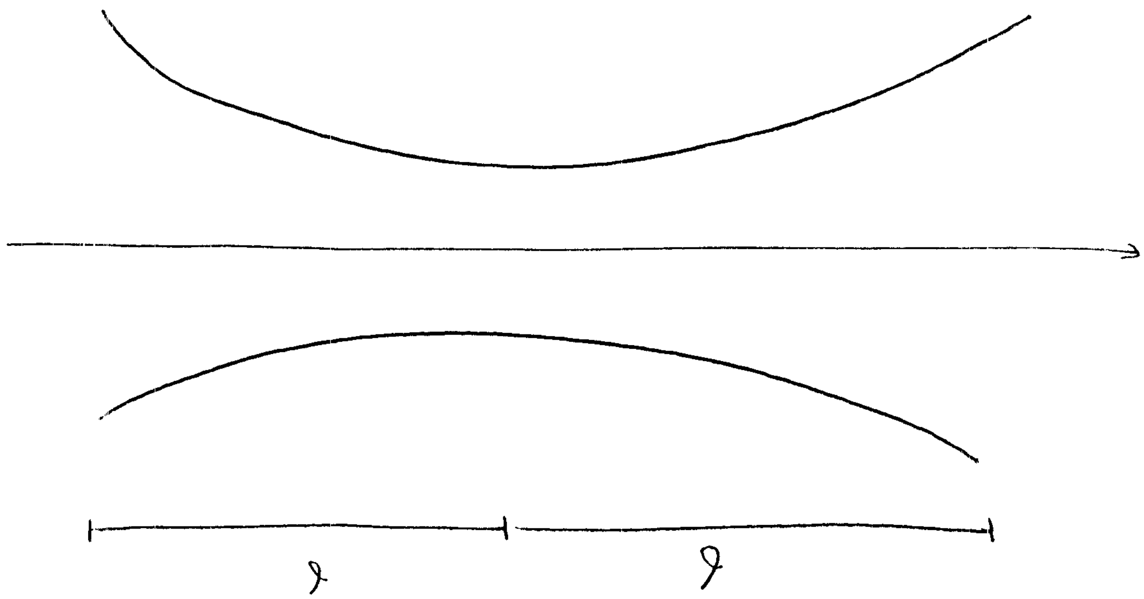
3 A thin square wafer has constant magnetization density $\vec{M} = M_0\hat{z}$. The wafer is ℓ long on each side and has thickness d. Compute the magnetic field at the center of the wafer.

4 A ring with radius a and constant linear charge density λ lies in the $x - y$ plane centered at the origin. Compute the electric field at a point R along the z-axis.

5 The potential on the surface of an infinite cylinder is given by $V(a, \phi) = V_0(\sin(\phi) + \cos(2\phi))$. Find the potential and the electric field at all points inside the cylinder.

6 Two spherical shells of radius a and b, $a < b$, have uniformly distributed charges $Q_a = Q$ and $Q_b = -Q$. Compute the energy between the shells.

1



Assume a current I flows through the wire.

The current density is

$$J = \frac{I}{A}$$

This is related to the field by

$$J = \sigma E$$

$$E = \frac{\sigma I}{A}$$

~~The potential difference along the wires is~~

$$\cancel{\Delta V = -\int_{-\ell}^{\ell} E\,dx = \sigma I \int_{-\ell}^{\ell} \frac{dx}{A_0 + x^2}}$$

The potential difference across the ends of the wire is given by

$$\Delta V = -\int_{-\ell}^{\ell} \vec{E} \cdot d\vec{l}$$

$$= -\sigma I \int_{-\ell}^{\ell} \frac{dx}{A(x)}$$

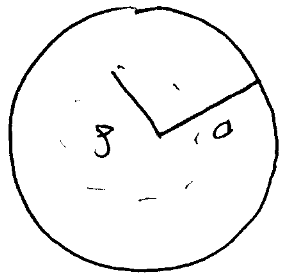
$$= \frac{-\sigma I}{A_0} \int_{-\ell}^{\ell} \frac{dx}{a^2+x^2}$$

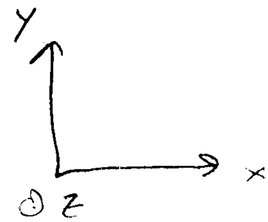
$$= \frac{-2\sigma I}{A_0} \int_{0}^{\ell} \frac{dx}{a^2+x^2} = \frac{-2\sigma I}{a A_0} \tan^{-1}\left(\frac{\ell}{a}\right)$$

The resistance is given by Ohm's law

$$R = \frac{\Delta V}{I} = \frac{2\sigma}{a A_0} \tan^{-1}(\ell/a)$$

(2)





The current flowing through a surface bounded by a circular Amperian path of radius ρ is

$$I_{enc} = \int_0^{\rho} 2\pi\rho \, d\rho \; J$$

$$= 2\pi J_0 \int_0^{\rho} \rho^3 \, d\rho$$

$$= \frac{\pi J_0}{2} \rho^4$$

If $\rho < a$ and

$$I_{enc} = \frac{\pi J_0}{2} a^4$$

if $\rho > a$.

The magnetic field is found from Ampere's Law,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$2\pi\rho B = \mu_0 I_{enc}$$

$$B = \frac{\mu_0 I_{enc}}{2\pi\rho}$$

For $\rho < a$,

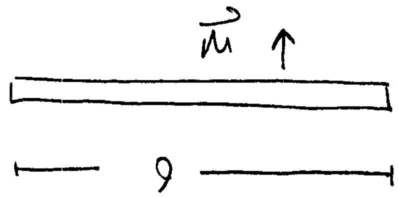
$$\vec{B} = \frac{\mu_0}{2\pi\rho}\left(\frac{\pi J_0}{2}\rho^4\right)\hat{\phi} = \frac{\mu_0 J_0 \rho^3}{4}\hat{\phi}$$

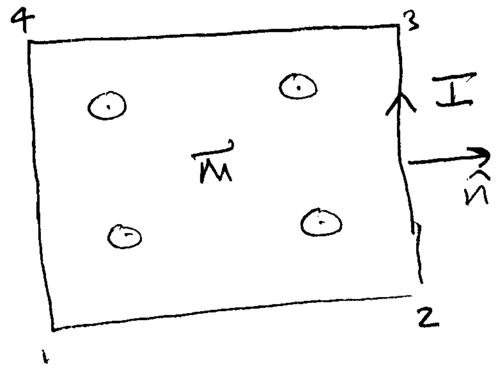
For $\rho > a$,

$$\vec{B} = \frac{\mu_0}{2\pi\rho}\left(\frac{\pi J_0 a^4}{2}\right) = \frac{\mu_0 J_0 a^4}{4\rho}\hat{\phi}$$

Direction found by Right Hand Rule.

3





The magnetization produces a surface current density

$$\vec{K}_b = \vec{M} \times \hat{n} = M_0 \hat{z} \times \hat{n}$$

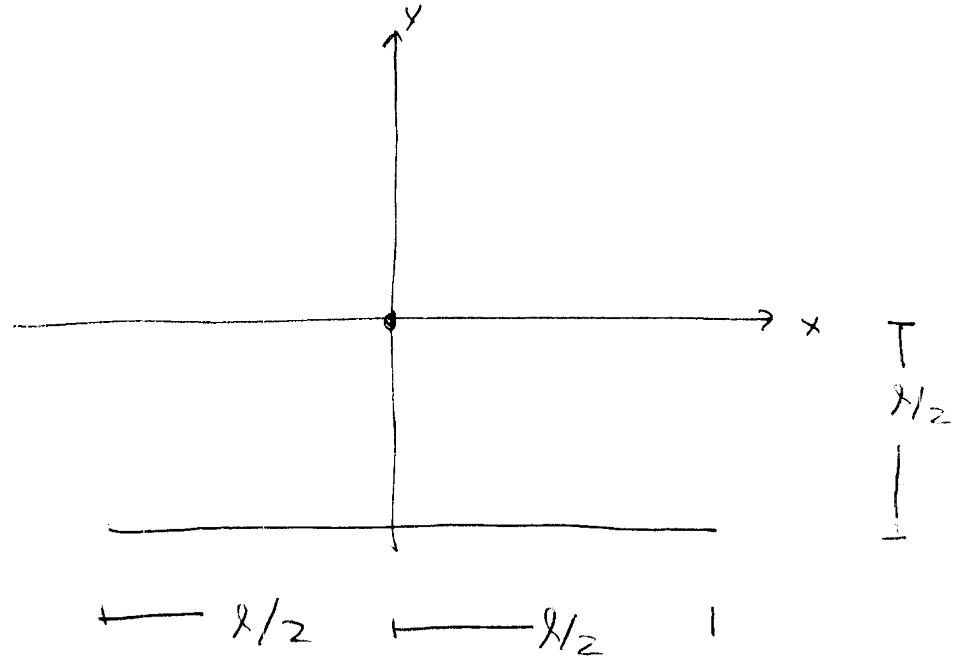
$$|\vec{K}_b| = M_0$$

Since the wafer is thin, this produce an effective current $I = |K_b| d = M_0 d$ around the outer edge.

The field of each edge is evidently the same, so the total field at the center is

$$\vec{B}_0 = 4 \vec{B}_{12}$$

The field from segment 1,2 is



The displacement vector from $\vec{r}_i = (x, -\ell/2, 0)$

to $\vec{r}_c = (0, 0, 0)$ is

$$\vec{r}\,'' = (-x, \ell/2, 0)$$

The current is $\vec{I} = I\, d\vec{\ell} = I\, dx\, \hat{x}$

and the required cross-product

$$\vec{I} \times \vec{r}\,'' = \frac{I\, dx\, \ell}{2}\, \hat{z}$$

Biot-Savart

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{d\vec{I}\times\vec{r}''}{r''^3}$$

$$= \frac{\mu_0}{4\pi}\int_{-\ell/2}^{\ell/2} \frac{I\ell\,dx\,\hat{z}}{2(x^2+(\ell/2)^2)^{3/2}}$$

$$= 2\cdot\frac{\mu_0 I\ell\,\hat{z}}{8\pi}\int_0^{\ell/2}\frac{dx}{(x^2+(\ell/2)^2)^{3/2}}$$

$$= \frac{\mu_0 I\ell}{4\pi}\hat{z}\left[\frac{x}{(\ell/2)^2\sqrt{x^2+(\ell/2)^2}}\right]_0^{\ell/2}$$

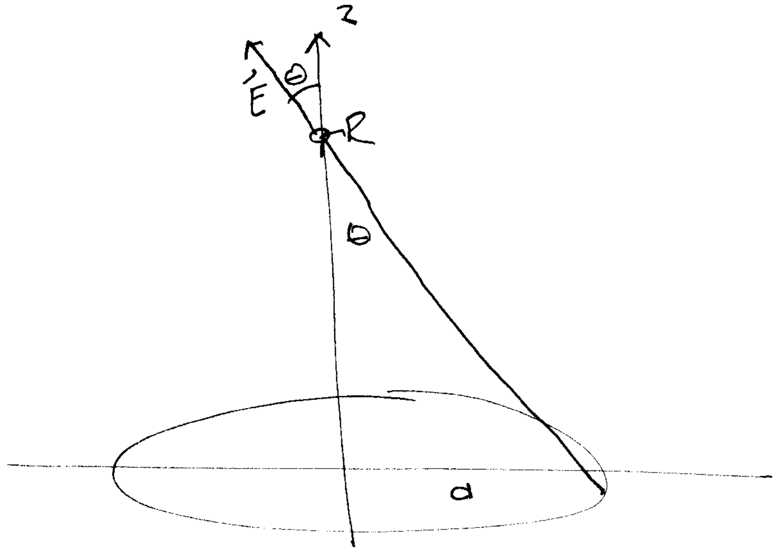
$$= \frac{\mu_0 I\ell}{4\pi}\hat{z}\left(\frac{\ell/2}{\sqrt{2}(\ell/2)^3}\right)$$

$$= \frac{\mu_0 I}{2\pi\sqrt{2}}\cdot\frac{1}{\ell/2}\hat{z} = \frac{\mu_0 I}{\sqrt{2}\pi\ell}\hat{z}$$

The total field is $4\vec{B}$

$$\vec{B}_0 = \frac{4\mu_0 I}{\sqrt{2}\pi\ell}\hat{z}$$

4



The field of a small chunk dq is

$$|d\vec{E}| = \frac{k\,dq}{R^2+a^2}$$

The z-components add. All other components cancel.

The z-component is

$$dE_z = \frac{k\,dq}{R^2+a^2}\cos\theta = \frac{k\,dq}{R^2+a^2}\,\frac{R}{\sqrt{R^2+a^2}}$$

The total field is the integral of this field

$$|\vec{E}| = \int \frac{Rk\,dq}{(R^2+a^2)^{3/2}} = \frac{kRQ}{(R^2+a^2)^{3/2}}$$

where $Q = 2\pi a\lambda$, the total charge of the ring.

$$\vec{E} = \frac{KR\,(2\pi a\lambda)}{(R^2+a^2)^{3/2}}\,\hat{z} = \frac{2\pi a\lambda KR}{(R^2+a^2)^{3/2}}$$

$$= \frac{Ra\lambda}{2\varepsilon_0\,(R+a^2)^{3/2}}\,\hat{z}$$

(5) The potential inside the ~~spherical~~ cylindrical system is given by

$$V_i(\rho, \phi) = \sum_n A_n \rho^n \cos n\phi + B_n \rho^n \sin n\phi$$

other terms are discarded because V_i must be finite.

At the boundary,

$$\cancel{V = V_0 (\cos\phi + \sin 2\phi)}$$

$$V = V_0 (\sin\phi + \cos 2\phi) = \sum_n A_n a^n \cos n\phi + B_n a^n \sin n\phi$$

By orthoguality,

$$V_0 = B_1 a \qquad V_0 = A_2 a^2$$

and the potential inside becomes

$$V_i(\rho, \phi) = V_0 \frac{\rho^2}{a^2} \cos 2\phi + V_0 \left(\frac{\rho}{a}\right) \sin\phi$$

The field is then

$$\vec{E}_i = -\nabla V = -\frac{\partial V}{\partial \rho}\hat{\rho} - \frac{1}{\rho}\frac{\partial V}{\partial \rho}\hat{\phi}$$

$$= -\left(\frac{2V_0\rho}{a^2}\cos 2\phi + \frac{V_0}{a}\sin\phi\right)\hat{\rho}$$

$$-\frac{1}{\rho}\left(-2\frac{V_0\rho^2}{a^2}\sin 2\phi + V_0\left(\rho/a\right)\cos\phi\right)\hat{\phi}$$

(6) The electric field between the shells is

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

The energy density is then

$$u = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0\left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^2$$

$$= \frac{Q^2}{32\pi^2\varepsilon_0 r^4}$$

Integrate the energy density over the volume between the spheres.

$$U = \int u\,dv = \int_a^b dr \int_0^{2\pi} \cancel{r\,d\theta}\, r\sin\theta\, d\phi \int_0^{\pi} r\,d\theta\, u$$

$$= 4\pi\int_a^b r^2 dr\, u = 4\pi\left(\frac{Q^2}{32\pi^2\varepsilon_0}\right)\int_a^b \frac{dr}{r^2}$$

$$= \frac{Q^2}{8\pi\varepsilon_0}\left(-\frac{1}{r}\right)\Bigg|_a^b = \frac{Q^2}{8\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

Could also get result from capacitance.