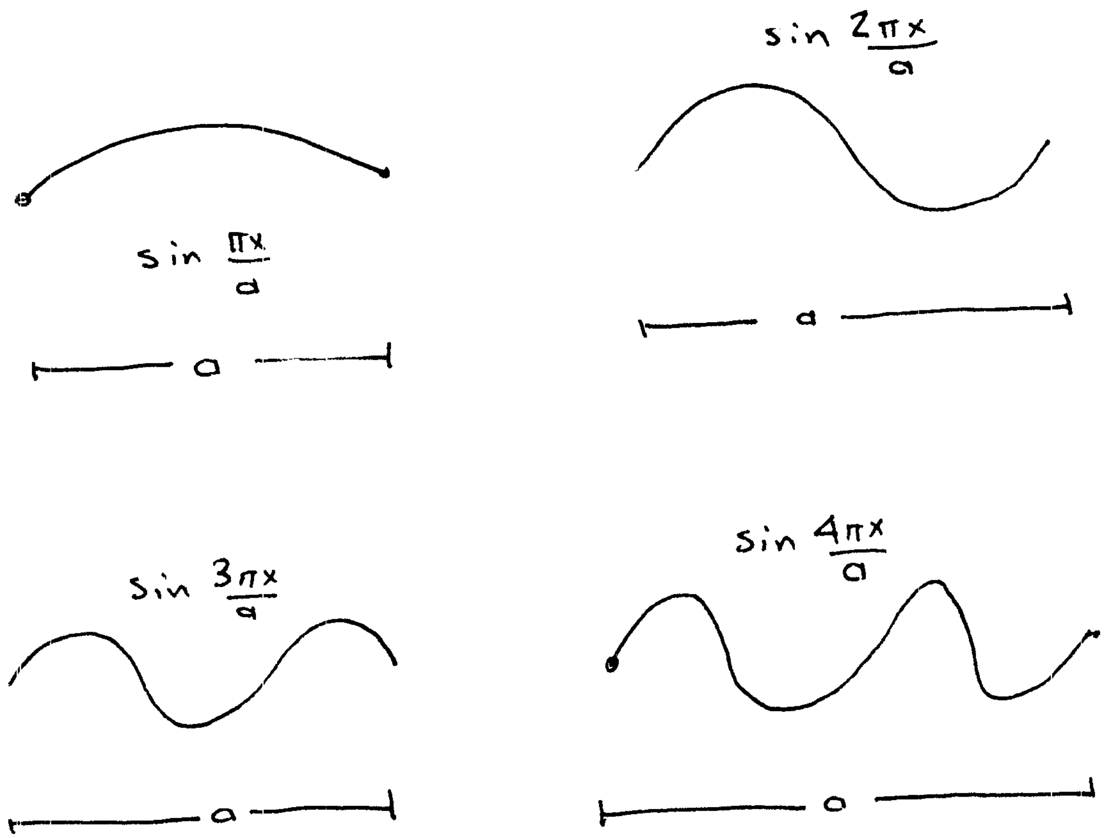
Intro to Fourier

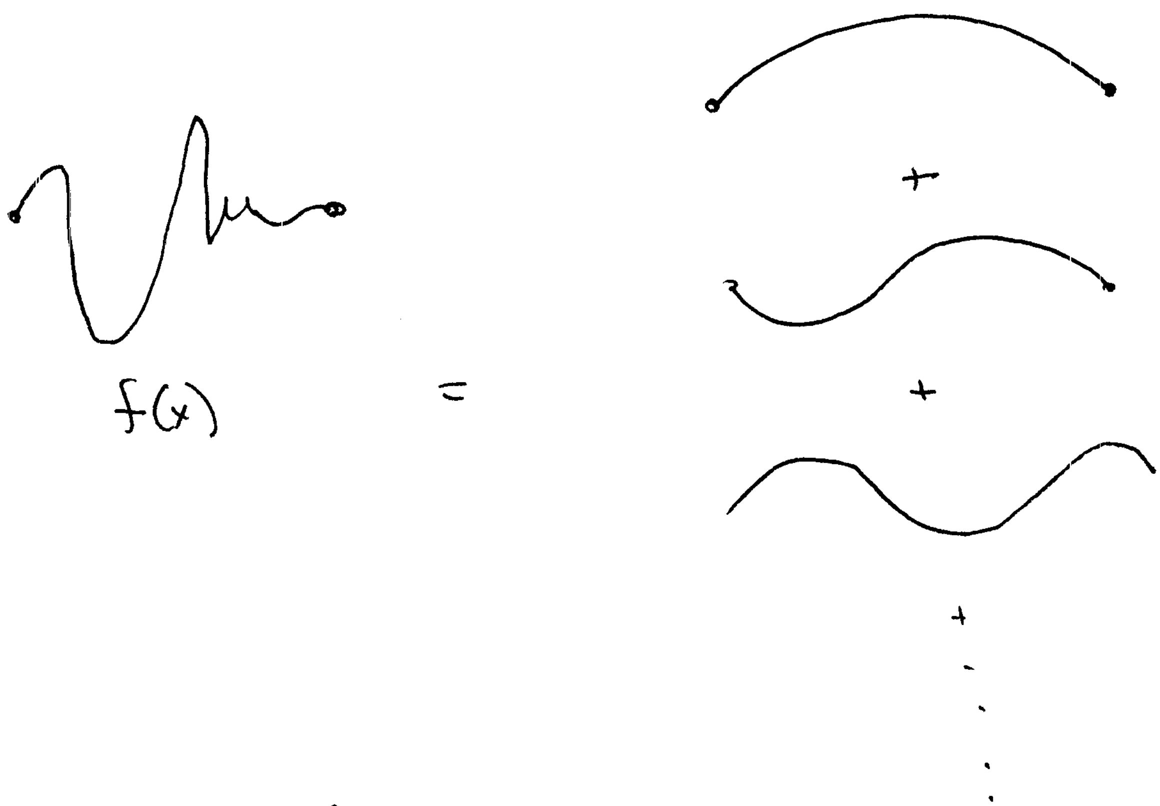
Physical systems have preferred states they like to be in. For example, consider a string stretched between two points. The string has some pure modes of vibration.



For some sets of equations and boundary conditions the preferred modes of the system ~~and~~ are <u>independent (orthogonal)</u> and <u>complete</u>.

What does this mean?

Completeness – We can make any arbitrary state of the system out the ~~preffe~~ preferred modes.



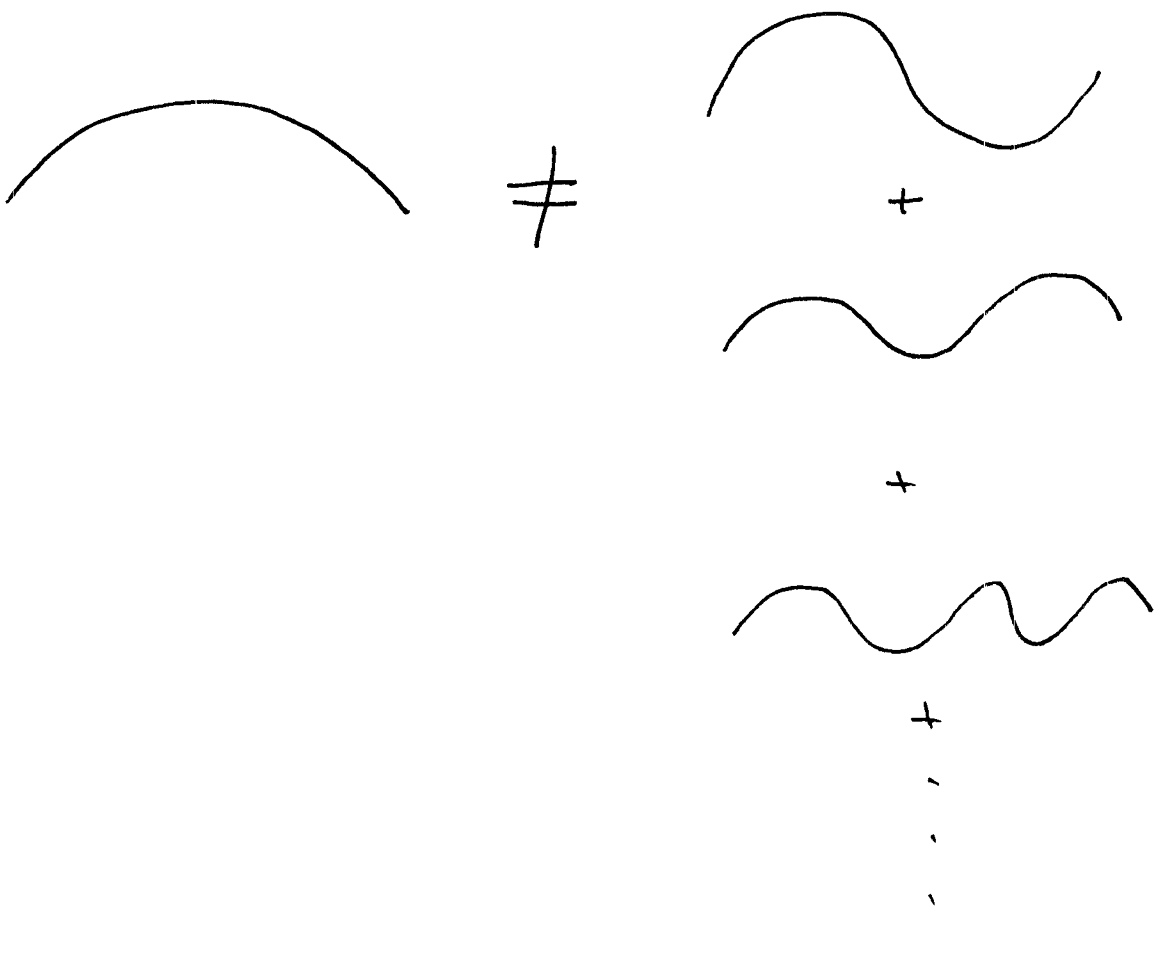
$$f(x) = \sum A_n \sin k_n x$$

where for our string system the preferred modes are

$$\sin k_n x = \sin \frac{n\pi x}{a}$$

$$k_n = \frac{n\pi}{a}$$

Independance (orthogonality) - We cannot make one of the pure modes out of some combination of the other pure modes.



or mathematically

$$\int_0^a \sin k_n x \sin k_m x \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{a}{2} & \text{if } m = n \end{cases}$$

We can also write this as,

$$\int_0^a \sin k_n x \sin k_m x \, dx = \frac{a}{2} \sigma_{mn} \qquad \text{(orthogonality)}$$

where $$\sigma_{mn} = \begin{cases} 0 & \text{if } m = n \\ 1 & m \neq n \end{cases}$$

Using this relation, we can calculate the coefficients A_n.

<u>Ex</u> Find the expansion of $x(\cancel{x}-a) = f(x)$ in a Fourier series ($\sin k_n x$).

<u>Sln</u>

$$f(x) = \sum_n A_n \sin k_n x = x^2 - ax$$

$$\sin k_m x \, f(x) = \sum_n A_n \sin k_m x \sin k_n x$$

$$\int_0^a \sin k_m x \, f(x) \, dx = \sum_n A_n \int_0^a \sin k_m x \sin k_n x \, dx$$

$$\int_0^a \sin k_m x \, f(x)\, dx = \begin{cases} \sum_n A_n \cdot 0 & m \neq n \\ \sum_n \frac{a}{2} A_m & m = n \end{cases}$$

$$= \frac{a}{2} A_m$$

$$A_m = \frac{2}{a} \int_0^a \sin k_m x \, f(x)\, dx$$

The above is general, now we apply it to our string where $f(x) = x^2 - ax$. This gets a bit messy but everything is done with Schaum's and involves no new idea, We just have to integrate. I will work some easier examples later on.

$$A_m = \frac{2}{d}\int_0^a \sin k_m x \left(x^2 - ax\right) dx$$

$$= \frac{2}{d}\int_0^a \sin k_m x \; x^2 \, dx \quad - 2\int_0^a \sin k_m x \; x \, dx$$

$$= \frac{2}{d}\left(\frac{2x}{k_m^2} \sin k_m x + \left(\frac{2}{k_m^3} - \frac{x^2}{k_m}\right) \cos k_m x \right)\Bigg|_0^a$$

$$- 2\left(\frac{\sin k_m x}{k_m^2} - \frac{x \cos k_m x}{k_m} \right)\Bigg|_0^a$$

$\sin k_m 0 = 0 \qquad \sin k_m a = \sin m\pi = 0$

$\cos k_m 0 = 1 \qquad \cos k_m a = \cos m\pi = (-1)^m$

$$A_m = \frac{2}{a}\left(\left(\frac{2}{k_m^3} - \frac{a^2}{k_m}\right)(-1)^m - \frac{2}{k_m^3}\right)$$

$$-2\left(- \frac{a(-1)^m}{k_m}\right)$$

$$A_m = \frac{4}{a k_m^3}(-1)^m - \frac{4}{a k_m^3} \underbrace{- \frac{2a}{k_m}(-1)^m + \frac{2a}{k_m}(-1)^m}_{0}$$

$$= \frac{4}{a k_m^3}\left[(-1)^m - 1\right]$$

If m = even, $A_m = 0$

If m = odd $[\ ] = -2$

$$A_m = \frac{-8}{a k_m^3} = \frac{-8}{a\left(\frac{m\pi}{a}\right)^3} = \frac{-8a^2}{\pi^3 m^3}$$

Finally, what does this mean

$$f(x) = x^2 - ax = \sum_{m\ odd} \frac{-8a^2}{\pi^3 m^3} \sin \frac{m\pi}{a} x$$

Let's find the first few terms

$$A_1 = \frac{-8a^2}{\pi^3} = \cancel{-3.7\times10^{-4}}\ -0.25 \quad \text{if } a=1$$

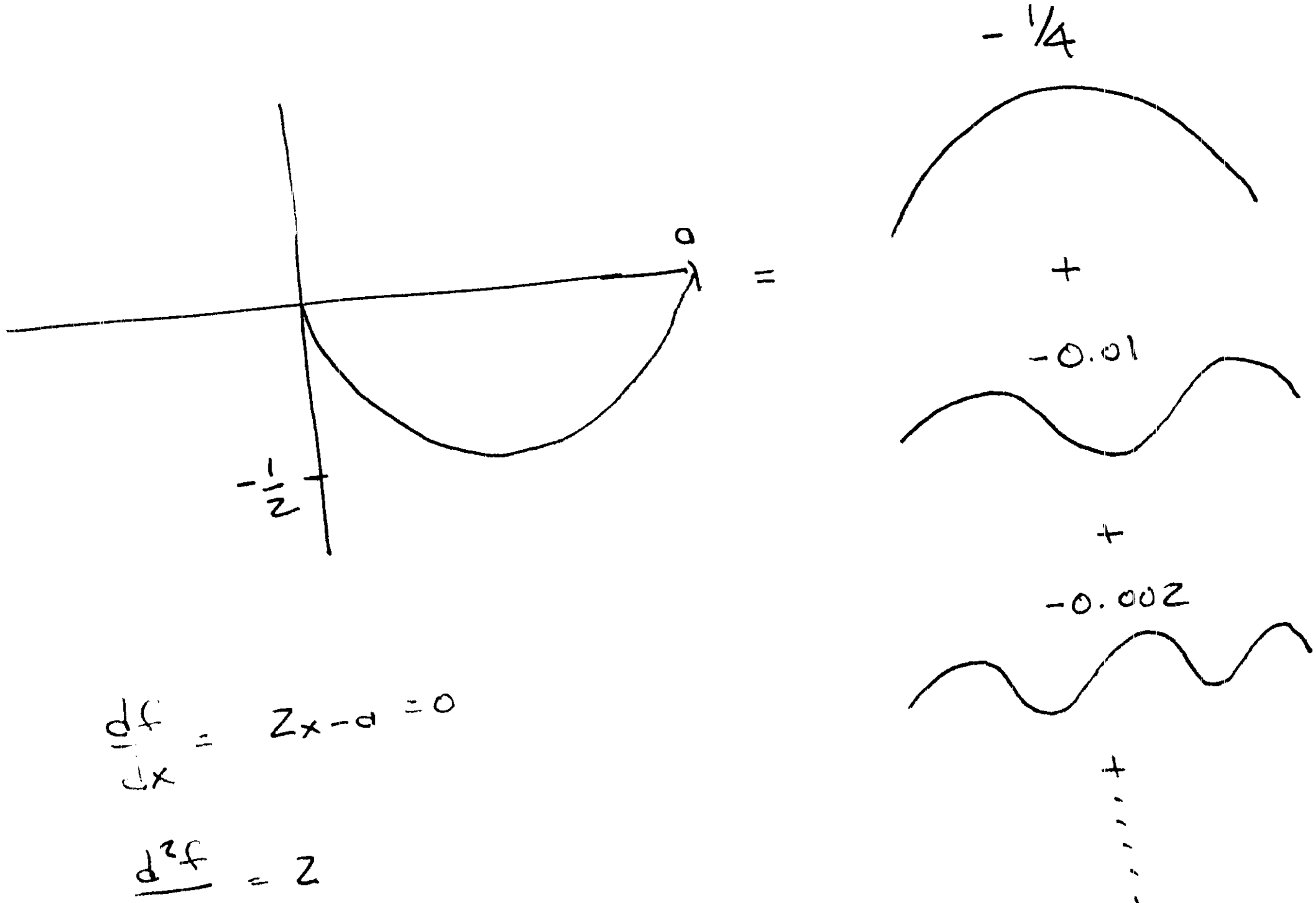
$$A_3 = \frac{-8a^2}{27\pi^3} = \cancel{-1.5\times10^{-4}}\ \cancel{-0.001}$$

$$= -0.01$$

$$A_5 = \frac{-8a^2}{125\pi^3} = \cancel{-3.3\times10^{-4}}$$

$$-0.002$$

So as we try to make a parabola out of sine waves



$$\frac{df}{dx} = 2x - a = 0$$

$$\frac{d^2 f}{dx^2} = 2$$

$$x_{min} = \frac{a}{2}$$

$$f(a/2) = \frac{a^2}{4} - \frac{a^2}{2} = -\frac{1}{2}a^2$$

So the parabola is mostly  flipped over with lots of other stuff to fix things up.