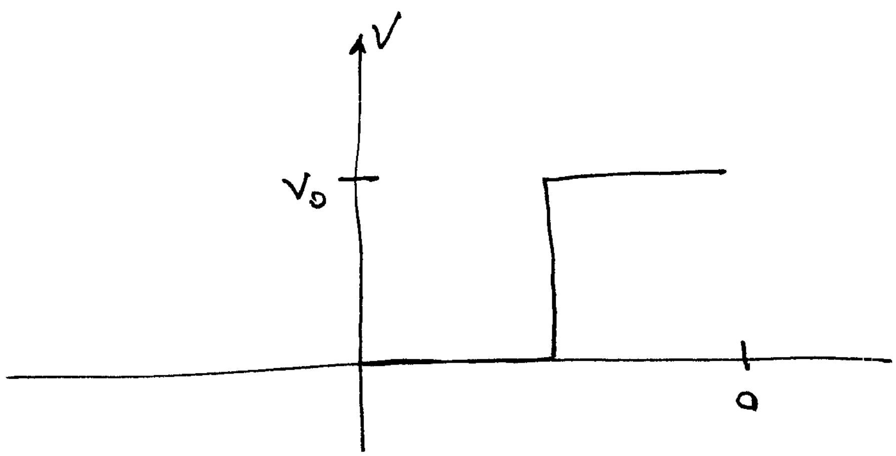
# Intro to Fourier

**Ex** Write Fourier Series for

$$V(x) = \begin{cases} 0 & 0 < x < a/2 \\ V_0 & \frac{a}{2} < x < a \end{cases}$$




## Complete Sets of Functions

(1) $\sin k_n x$ $\quad k_n = \frac{n\pi}{a}$ $\quad$ Range $0, a$

(2) $\cos k_n x$ $\quad k_n = \frac{n\pi}{a}$ $\quad$ Range $0, a$

(3) $\sin k_n x,\ \cos k_n x$ $\quad k_n = \frac{2n\pi}{a}$ $\quad$ Range $-a, a$

We will continue to use the sine expansion

$$V(x) = \sum_n A_n \sin k_n x \qquad k_n = \frac{n\pi}{a}$$

Orthogonality

$$\int_0^a \sin k_m x \sin k_n x \, dx = \frac{a}{2} \delta_{nm}$$

Solve for $A_m$

$$\int_0^a V(x) \sin k_m x \, dx = \sum_n A_n \int_0^a \sin k_n x \sin k_m x \, dx$$

$$= \frac{a}{2} A_m$$

$$A_m = \frac{2}{a} \int_0^a V(x) \sin k_m x \, dx$$

$$A_m = \frac{2}{a}\int_0^a V(x) \sin k_m x \, dx$$

$$= \frac{2V_0}{a}\int_{a/2}^a \sin k_m x \, dx$$

$$= \frac{-2V_0}{a k_m} \cos k_m x \Big|_{a/2}^{a} = \frac{-2V_0}{a k_m}\left(\cos m\pi - \cos \frac{m\pi}{2}\right)$$

$$\cos m\pi = (-1)^m \qquad \cos \frac{m\pi}{2} = \begin{cases} 0 & m \text{ odd} \\ (-1)^{m/2} & m \text{ even} \end{cases}$$

| $m$ | $\cos m\pi$ | $\cos \frac{m\pi}{2}$ | $\cos m\pi - \cos \frac{m\pi}{2}$ |
|---|---|---|---|
| 1 | $-1$ | 0 | $-1$ |
| 2 | 1 | $-1$ | 2 |
| 3 | $-1$ | 0 | $-1$ |
| 4 | 1 | 1 | 0 |
| 5 | $-1$ | 0 | $-1$ |

$$A_m = \begin{cases} \dfrac{2V_0}{a k_m} & m \text{ odd} \\[2ex] \dfrac{2V_0}{a k_m}\left(1-(-1)^{m/2}\right) & m \text{ even} \end{cases}$$

At which point we have done what we can.

$$V(x) = \sum_m A_m \sin k_m y$$

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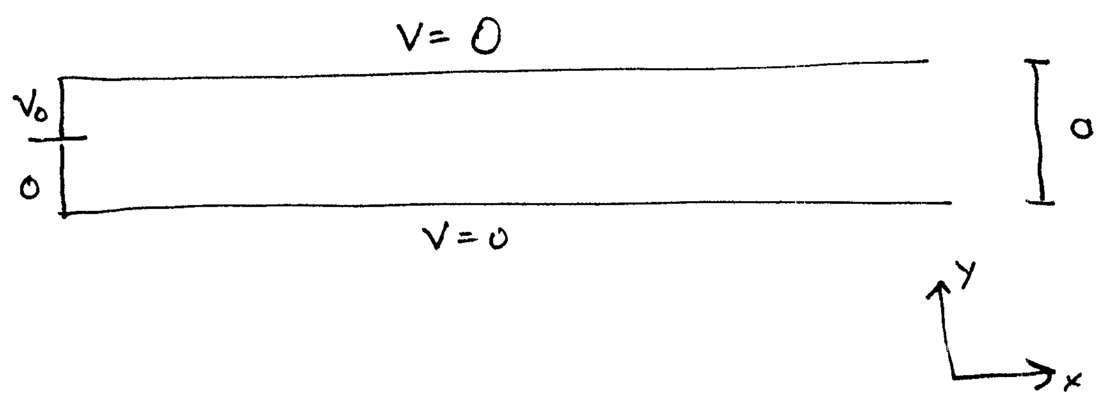
The above solution immediately suggests a boundary value problem.



Method

(1) Select correct version of Laplace's Eqn

(2) Identify all solutions.

(3) Determine boundary conditions

(4) Fix coefficients of solutions based on boundary conditions.

Step (1) Laplace Eqn for System

$$\nabla^2 V = 0 = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Step (2) Propose Separated Solution

$$V = X(x)\,Y(y)$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

I would like the y-solutions to oscillated

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{= k^2} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{= -k^2} = 0$$

$$k^2 - k^2 = 0$$

$$\frac{d^2X}{dx^2} - k^2 X = 0 \qquad X = e^{\pm kx}$$

$$\frac{d^2Y}{dy^2} + k^2 Y = 0 \qquad Y = \sin ky, \cos ky$$

So the solution to the system looks like

$$V(x,y) = \left[e^{kx}, e^{-kx}\right] \times \left[\cos ky, \sin ky\right]$$

* Note, this could have been written directly.

Step (3) ~~Apply Boundary~~ Determine Boundary Conditions

(A) $V(x,y) \to 0$ at $x \to \infty$

(B) $V(x,0) = 0$ (Bottom at $V=0$)

$V(x,a) = 0$ (Top at $V=0$)

(C) $V(0,y) = \begin{cases} V_0 & a/2 < y < a \\ 0 & 0 < y < a/2 \end{cases}$

Step (4) - Apply Boundary Conditions to Determine Coefficients.

The boundary conditions will do 3 things

(1) Eliminate some solutions

(2) Select a discrete set of values for $k$

$$k \to k_n$$

(3) Set values for the coefficients $A_n, B_n \ldots$ in the series.

It is best if the boundary conditions are applied so these effects happen in the above order.

Apply Boundary Conditions

(A) $V \to 0$ as $x \to \infty$

$\Rightarrow$ Cannot have $e^{+kx}$ solutions

(B) $V(x,0) = 0 \quad V(x,a) = 0$

$\Rightarrow$ Cannot have $\cos ky$ solutions because $\cos 0 = 1$

$\Rightarrow \quad k \to k_n = \dfrac{n\pi}{a}$ because $\sin k_n a = 0$

(c) Our solution at this point looks like

$$V(x,y) = \sum_{n=1}^{\infty} A_n e^{-k_n x} \sin k_n y$$

The final boundary condition is $V(0,y) = \begin{cases} V_0 & a/2 < y < a \\ 0 & y < a/2 \end{cases}$

Select $A_n$ to satisfy this condition

$$V(0,y) = \sum A_n \sin k_n y = \begin{cases} V_0 & a/2 < y < a \\ 0 & 0 < y < a/2 \end{cases}$$

Use Orthogonality

$$\int_0^a \sin k_m y \, V(0,y) = \sum A_n \int_0^a \sin k_m y \sin k_n y \, dy$$

$$= \frac{a}{2} A_m$$

$$A_m = \frac{2}{a} \int_0^a \sin k_m y \, V(0,y) dy$$

$$= \frac{2V_0}{a} \int_{a/2}^a \sin k_m y \, dy = \frac{-2V_0}{a k_m} \left( \cos m\pi - \cos \frac{m\pi}{2} \right)$$

As before

$$\cos m\pi = (-1)^m$$

$$\cos \frac{m\pi}{2} = \begin{cases} 0 & m \text{ ~~even~~ odd} \\ (-1)^{m/2} & m \text{ even} \end{cases}$$

$$A_m = \begin{cases} \dfrac{2V_0}{a k_m} & m \text{ odd} \\[2ex] \dfrac{2V_0}{a k_m}\left(1 - (-1)^{m/2}\right) & m \text{ even} \end{cases}$$

$$V(x,y) = \sum A_m e^{-k_m x} \sin k_m y$$

At this point I could ask you to evaluate some of the coefficients, write a series for the field or a series for the charge density on one of the plates.