PHYS 3414 - Electricity and Magnetism- Homework Set 1

Cal III (Chapter 1) and UPII Review

Due 12:30pm Wednesday January 23, 2007 at the beginning of class.

Good's Problems

Office Hours
4:30-5:30
T

1.12

1.16

1.18

1.20

1.24

1.26 You must do this using one of the forms of Stoke's Thm. Note: $(d\vec{a} \times \nabla) \times \vec{r} \neq d\vec{a} \times (\nabla \times \vec{r})$

Additional Problems

Problem A1 This problem takes you through the calculation of the electric field of a finite cylinder with uniform volume change density ρ through a sequence of simpler calculations.

(a) Compute the electric field along the axis of a ring of charge with charge density λ and radius R.

(b) Use the result of this calculation to calculate the field along the axis of a disk with uniform surface charge density σ and radius R.

(c) Check your calculation by showing your result becomes the field of an infinite plane in the limit $R \to \infty$.

(d) Use the result in (b) to calculate the electric field along the axis outside of a cylinder of uniform volume charge density ρ and radius R. The cylinder occupies the region $-L < x < L$ along the x-axis.

Problem A2 Consider the unit vectors that form cylindrical coordinates $(\hat{\rho}, \hat{\phi}, \hat{z})$ and spherical coordinates $(\hat{r}, \hat{\phi}, \hat{\theta})$.

(a) Draw the unit vectors for an arbitrary point in Cartesian coordinates for each system of coordinates.

(b) Express each unit vector in terms of $x, y, z, \hat{x}, \hat{y}, \hat{z}$.

(c) Using a set of vectors, draw the fields $\vec{E} = \gamma\hat{\rho}$, $\vec{E} = \gamma\hat{\phi}$, $\vec{E} = \gamma\hat{r}$, and $\vec{E} = \gamma\hat{\theta}$.

(d) For which of the above, do you expect $\nabla \cdot \vec{E} \neq 0$ or $\nabla \times \vec{E} \neq 0$? Why?

(e) Compute $\nabla \cdot \vec{E}$ and $\nabla \times \vec{E}$ for each field in part (c).

(f) Compute $\nabla \cdot \vec{E}$ and $\nabla \times \vec{E}$ for $\vec{E} = \gamma\hat{\phi}$ and $\vec{E} = \gamma\hat{r}$ in Cartesian coordinates.

UPII Problems

Problem A3 **(a)** A sphere containing a uniform volume charge density, $\rho = 100.0\mu\text{C/m}^3$, has radius 5.0cm. Compute the total charge of the sphere.

(b) A cylinder with length 0.20m and diameter 6.0cm has uniform surface charge, $\sigma = 10.0\mu\text{C/m}^2$. Compute the total charge of the cylinder, excluding the ends.

Problem A4 In class you compared the strength of the electric force on an electron due to a proton to the strength of the gravitational force on an electron due to a proton. Which of the following best describes the relationship?

Select One of the Following:

(a) The forces have about the same strength

(b) Gravity is a little stronger

(c) The electric force is a little stronger

(d) Gravity is a lot stronger

(e) The electric force is a lot stronger

Problem A5 On the first day of class, I computed the energy that would be liberated if all the electrons disappeared from block of carbon. In this problem, we evaluate how much force would be required to separate the electrons from the protons. The protons in a 2oz block of carbon have charge 2×10^6C and the electrons have charge -2×10^6C. Suppose the protons and electrons are separated by a distance of 10cm. What is the magnitude of the attractive force between the ball of protons and the ball of electrons if they are treated as point charges?

Total at top ~~65/70~~ Monday Morning

1.12 (a)	10 pts
1.16	10 pts
1.20	20 pts
1.24	20 pts
A1 (a)	10 pts —
A2 (e)	24 pts
A4	5 pts
	1 bonus
	100

Office Hours Friday 10:30 - 11:30 F

SCEN 110

1.12 (a)

$$\nabla\cdot(f\vec{A}) = \frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z)$$

$$= A_x\frac{\partial f}{\partial x} + f\frac{\partial A_x}{\partial x} + A_y\frac{\partial f}{\partial y} + f\frac{\partial A_y}{\partial y}$$

$$+ A_z\frac{\partial f}{\partial z} + f\frac{\partial A_z}{\partial z}$$

$$= \vec{A}\cdot\nabla f + f\,\nabla\cdot\vec{A}$$

Part (b)

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_yB_z - A_zB_y)\hat{x} - (A_xB_z - A_zB_x)\hat{y}$$

$$+ (A_xB_y - A_yB_x)\hat{z}$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = \frac{\partial}{\partial x}\left(A_y B_z - A_z B_y\right) - \frac{\partial}{\partial y}\left(A_x B_z - A_z B_y\right) + \frac{\partial}{\partial z}\left(A_x B_y - A_y B_x\right)$$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}$$

$$\vec{B} \cdot (\nabla \times \vec{A}) = B_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - B_y\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + B_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Likewise

$$\vec{A}\cdot(\nabla\times\vec{B}) = A_x\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) - A_y\left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right)$$

$$+ A_z\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

If one then expands $\nabla\cdot(\vec{A}\times\vec{B})$ and compares terms with $\vec{A}\cdot(\nabla\times\vec{B}) + \vec{B}\cdot(\nabla\times\vec{A})$ they agree.

Part (c)

$$\nabla\times(\vec{A}\times\vec{B}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_yB_z - A_zB_y & -(A_xB_z - A_zB_x) & (A_xB_y - A_yB_x) \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(A_xB_y - A_yB_x) + \frac{\partial}{\partial z}(A_xB_z - A_zB_x)\right]\hat{x}$$

$$- \left[\frac{\partial}{\partial x}(A_xB_y - A_yB_x) - \frac{\partial}{\partial z}(A_yB_z - A_zB_x)\right]\hat{y}$$

$$+ \left[-\frac{\partial}{\partial x}(A_xB_z - A_zB_x) - \frac{\partial}{\partial y}(A_yB_z - A_zB_y\right]\hat{z}$$

I will demostrate this for x term, and leave it at that.

$$(\vec{B}\cdot\nabla)\vec{A} = \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)\vec{A}$$

$$= \left[B_x\frac{\partial A_x}{\partial x} + B_y\frac{\partial A_x}{\partial y} + B_z\frac{\partial A_x}{\partial z}\right]\hat{x}$$

$$+ \left[B_x\frac{\partial A_y}{\partial x} + B_y\frac{\partial A_y}{\partial y} + B_y\frac{\partial A_z}{\partial y}\right]\hat{y}$$

$$+ \left[B_x\frac{\partial A_z}{\partial x} + B_y\frac{\partial A_z}{\partial y} + B_z\frac{\partial A_z}{\partial z}\right]\hat{z}$$

with a similar expression for $(\vec{A}\cdot\nabla)\vec{B}$

$$\vec{A}(\nabla\cdot\vec{B}) = \left(A_x\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right)\right)\hat{x}$$

$$+ \left(A_y\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right)\right)\hat{y}$$

$$+ \left(A_z\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right)\right)\hat{z}$$

with a similar expression for $\vec{B}(\nabla\cdot\vec{A})$

The x-component of

$$(\vec{B}\cdot\nabla)\vec{A} = (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot B) - B(\nabla\cdot\vec{A})$$

$$\cancel{B_x \frac{\partial A_x}{\partial x}} + B_y \frac{\partial A_x}{\partial y} + B_z \frac{\partial A_x}{\partial z}$$

$$-\left(\cancel{A_x \frac{\partial B_x}{\partial x}} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z}\right)$$

$$+\left(\cancel{A_x \frac{\partial B_x}{\partial x}} + A_x \frac{\partial B_y}{\partial y} + A_x \frac{\partial B_z}{\partial z}\right)$$

$$-\left(\cancel{B_x \frac{\partial A_x}{\partial x}} + B_x \frac{\partial A_y}{\partial y} + B_x \frac{\partial A_z}{\partial z}\right)$$

$$\stackrel{?}{=} A_x \frac{\partial B_y}{\partial y} + B_y \frac{\partial A_x}{\partial y} - A_y \frac{\partial B_x}{\partial y} - B_x \frac{\partial A_y}{\partial y}$$

$$+ A_x \frac{\partial B_z}{\partial z} + B_z \frac{\partial A_x}{\partial z} - A_z \frac{\partial B_x}{\partial z} - B_x \frac{\partial A_z}{\partial z}$$

All terms accounted for.

1b

$$f = x - xy + 3z^2$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= 0 + 0 + 6 = 6$$

1.18 $\vec{A} = x\hat{x} - xy\hat{y} + 3z^2\hat{z}$

By direct calculation,

$$\nabla^2\vec{A} = 6\hat{z}$$

Eqn 1.25 $\nabla^2\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla\times(\nabla\times\vec{A})$

$$\nabla\cdot\vec{A} = 1 - x + 6z$$

$$\nabla(\nabla\cdot\vec{A}) = -\hat{x} + 6\hat{z}$$

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -xy & 3z^2 \end{vmatrix}$$

$$= (0-0)\hat{x} - (0-0)\hat{y} + (-y)\hat{z}$$

$$= -y\hat{z}$$

$$\nabla \times (\nabla \times \vec{A}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & -y \end{vmatrix}$$

$$= (-1 - 0)\,\hat{x} - 0\,\hat{y} + 0\,\hat{z}$$

$$= -\hat{x}$$

$$\nabla^2 \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla \times (\nabla \times \vec{A})$$

$$= -\hat{x} + 6\hat{z} - (-\hat{x}) = 6\hat{z}$$

1.20 $\nabla \times \rho^n \hat{\phi} = ?$

$\Rightarrow$ Cylindrical coordinates is implied by the ρ

$$\vec{A} = 0\hat{\rho} + \rho^n \hat{\phi} + 0\hat{z}$$

$$A_\rho = 0 \qquad A_\phi = \rho^n \qquad A_z = 0$$

$$\nabla \times \vec{A} = \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \hat{\rho}$$

$$+ \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \hat{\phi}$$

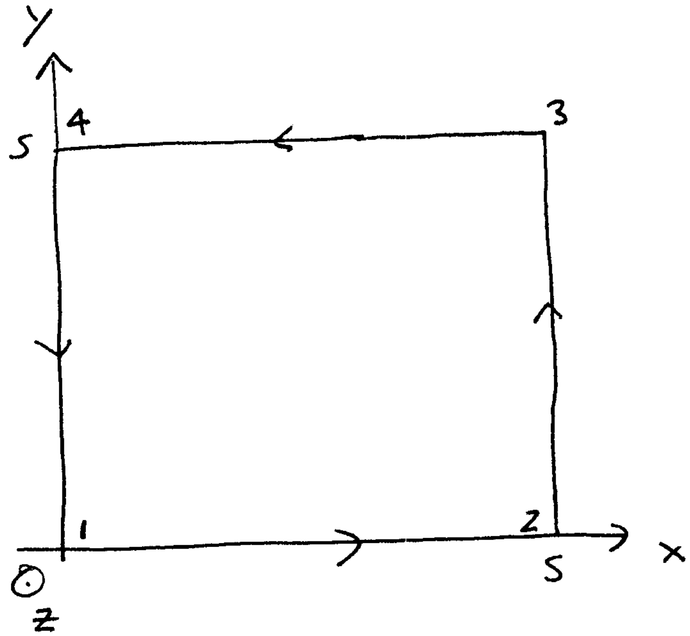
$$+ \frac{1}{\rho} \left(\frac{\partial (\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi} \right) \hat{z}$$

$$= (0 - 0)\hat{\rho} + (0 - 0)\hat{\phi}$$

$$+ \left(\frac{1}{\rho} \left(\frac{\partial \rho^{n+1}}{\partial \rho} \right) - 0 \right) \hat{z}$$

$$= \frac{1}{\rho}(n+1)\rho^n \hat{z} = (n+1)\rho^{n-1}\hat{z} \neq 0 \text{ unless } n = -1$$

1.24



Positive normal up.

Stoke's Thm

$$\oint_C \vec{A}\cdot d\vec{l} = \int_{1\to 2} + \int_{2\to 3} + \int_{3\to 4} + \int_{4\to 1}$$

$$\int_{1\to 2} \vec{A}\cdot d\vec{l} = \int_0^s x\,\hat{y}\cdot\hat{x}\,dx = 0 \qquad d\vec{l} = \hat{x}\,dx$$

$$\int_{2\to 3} \vec{A}\cdot d\vec{l} = \int_0^s x\,\hat{y}\cdot\hat{y}\,dy = s^2 \qquad d\vec{l} = \hat{y}\,dy, \quad x = s$$

$$\int_{3\to 4} \vec{A}\cdot d\vec{l} = \int_0^s x\,\hat{y}\cdot(-\hat{x}\,dx) = 0$$

$$\int_{4\to 1} \vec{A}\cdot d\vec{l} = \int_0^s x\,\hat{y}\cdot(-\hat{y}\,dy) = 0 \qquad d\vec{l} = -\hat{y}\,dy, \quad x = 0$$

$$\oint_C \vec{A}\cdot d\vec{l} = s^2$$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x & 0 \end{vmatrix}$$

$$= 0\hat{x} - 0\hat{y} + 1\hat{z}$$

$$\int \nabla \times \vec{A} \cdot d\vec{a} = \int_S \hat{z} \cdot \hat{z} \, da \qquad d\vec{a} = \hat{z}\, da$$

$$= S^2 \quad \checkmark$$

1.26

$$\oint_C \vec{r} \times d\vec{l} = -\int_S (d\vec{a} \times \nabla) \times \vec{r}$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$d\vec{a} = da_x\,\hat{x} + da_y\,\hat{y} + da_z\,\hat{z} = \hat{n}\,da$$

$$d\vec{a} \times \nabla = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ da_x & da_y & da_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$= \left(da_y \frac{\partial}{\partial z} - da_z \frac{\partial}{\partial y} \right) \hat{x}$$

$$- \left(da_x \frac{\partial}{\partial z} - da_z \frac{\partial}{\partial x} \right) \hat{y}$$

$$+ \left(da_x \frac{\partial}{\partial y} - da_y \frac{\partial}{\partial z} \right) \hat{z}$$

$$\cancel{(d\vec{a} \times \nabla) \times \vec{r} =}$$

$$(d\vec{a} \times \nabla) \times \vec{r} =$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ da_y \frac{\partial}{\partial z} - da_z \frac{\partial}{\partial y} & da_z \frac{\partial}{\partial x} - da_x \frac{\partial}{\partial z} & da_x \frac{\partial}{\partial y} - da_y \frac{\partial}{\partial x} \\ x & y & z \end{vmatrix}$$

$$= \hat{x}\left(-da_x - da_x\right) - \hat{y}\left(da_y + da_y\right) + \hat{z}\left(-da_z - da_z\right)$$

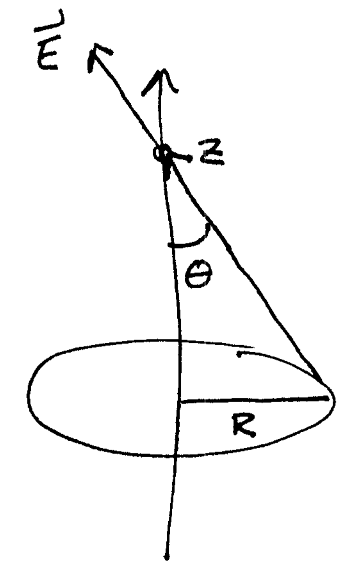
$$= -2\left(da_x \hat{x} + da_y \hat{y} + da_z \hat{z}\right) = -2 d\vec{a}$$

For a plane curve

$$-2\int_S d\vec{a} = -2\vec{a} = \oint_C \vec{r} \times d\vec{l}$$

$$\vec{a} = -\frac{1}{2}\oint_C \vec{r} \times d\vec{l}$$

A1



(a)

All charge is equidistant from z. The x-y component of the field cancels, so

$$\vec{E}(z) = \frac{kq}{(z^2+R^2)} \cos\theta \, \hat{z}$$

$$\cos\theta = \frac{z}{\sqrt{z^2+R^2}}$$

$$\vec{E}(z) = \frac{kqz}{(z^2+R^2)^{3/2}} \hat{z} \qquad q = 2\pi R \lambda$$

(b) The field of the disk is the sum of the field of a set of rings of thickness dr.

$$dq = \cancel{2\pi R \sigma dr} \; 2\pi r \sigma dr$$

$$\vec{E}(z) = \int_0^R \vec{E}_{ring} \, dr$$

$$\vec{E}_{\text{disk}}(z) = k\hat{z}\int_0^R \frac{z\,dq}{(z^2+R^2)^{3/2}}$$

$$= k\hat{z}\int_0^R \frac{z\,2\pi r\sigma\,dr}{(z^2+R^2)^{3/2}}$$

$$= 2\pi\sigma k z\,\hat{z}\int_0^R \frac{r\,dr}{(z^2+r^2)^{3/2}}$$

$$= \frac{\cancel{z}\,\sigma z\hat{z}}{2\varepsilon_0}\int_0^R \frac{r\,dr}{(z^2+r^2)^{3/2}}$$

<u>Do integral</u>

$$\int_0^R \frac{r\,dr}{(z^2+r^2)^{3/2}} = -\frac{1}{\sqrt{z^2+r^2}}\Bigg|_0^R$$

$$= \frac{1}{\sqrt{z^2}} - \frac{1}{\sqrt{R^2+z^2}}$$

$$\vec{E}_{\text{disk}} = \frac{\sigma z\hat{z}}{2\varepsilon_0}\left(\frac{1}{|z|} - \frac{1}{\sqrt{R^2+z^2}}\right)$$

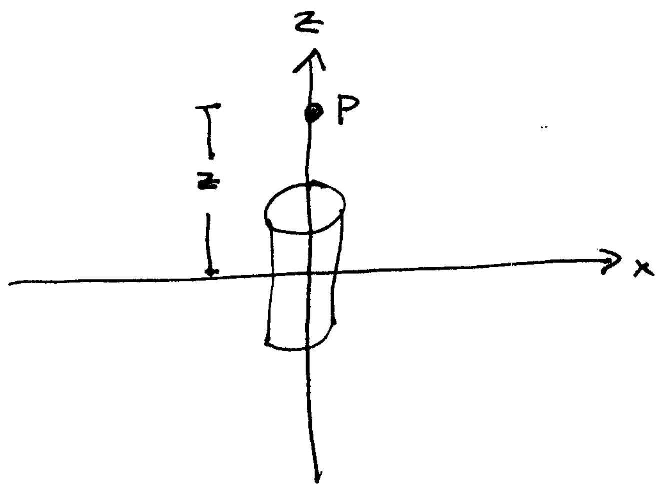
$$= \frac{\sigma}{2\varepsilon_0}\hat{z}\,\text{sign}(z) - \frac{\sigma}{2\varepsilon_0}\left(\frac{z}{\sqrt{R^2+z^2}}\right)\hat{z}$$

(c) If the disk becomes large, $R \to \infty$

$$\vec{E}_{disk} = \frac{\sigma}{2\varepsilon_0} \hat{z}\, \mathrm{sign}(z)$$ which is the

field of an infinite plane.

(d)



assume $z > 0$

$z - z' \geq 0$

$$\vec{E}_p(z) = \int_{-L}^{L} \vec{E}_{disk}(z')\, dz'$$

$$\vec{E}_{disk} = \frac{\sigma \hat{z}}{2\varepsilon_0} - \frac{\sigma}{2\varepsilon_0} \hat{z} \left(\frac{z - z'}{\sqrt{R^2 + (z-z')^2}} \right)$$

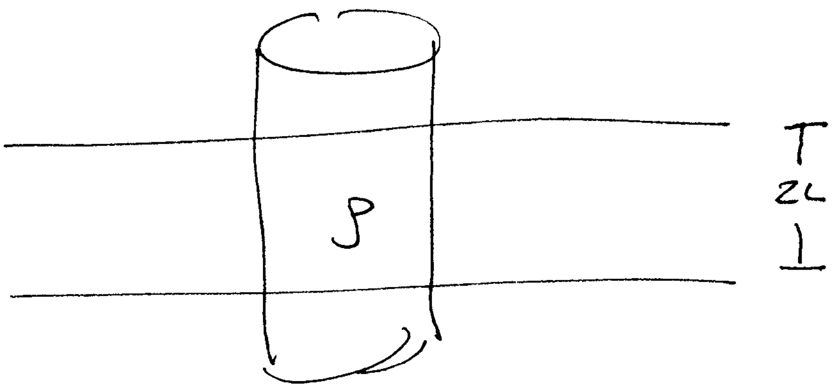
$$\sigma = \rho\, dz'$$

$$\vec{E}_p(z) = \frac{\rho \hat{z}}{2\varepsilon_0} \int_{-L}^{L} \left(dz' - \frac{z - z'}{\sqrt{R^2 + (z-z')^2}} \right) dz'$$

$$= \frac{\rho L \hat{z}}{\varepsilon_0} - \frac{\rho \hat{z}}{2\varepsilon_0} \int_{-L}^{L} \frac{z - z'}{(R^2 + (z-z')^2)^{1/2}}\, dz'$$

$$\vec{E}_\rho(z) = \frac{\rho L}{\epsilon_0}\hat{z} - \frac{\rho\hat{z}}{2\epsilon_0}\left(\sqrt{R^2+z^2+2zL+L^2} - \sqrt{R^2+z^2-2zL+L^2}\right)$$

Limits $R \to \infty$,



$$E_{top}A - E_{bottom}A = \frac{A(2L)\rho}{\epsilon_0} \qquad E_{top} = -E_{bottom}$$

$$\vec{E}_{top} = \frac{L\rho}{\epsilon_0}\hat{z} \quad \checkmark$$

$z \gg R, L$

$$\sqrt{R^2+z^2+2zL+L^2} - \sqrt{R^2+z^2-2zL+L^2}$$

$$= z\left(\sqrt{1+\frac{R^2+L^2}{z^2}+\frac{2L}{z}} - \sqrt{1+\frac{R^2+L^2}{z^2}-\frac{2L}{z}}\right)$$

Binomial Expansion $\sqrt{1+x} = 1 - \frac{1}{2}x + \cdots$

$$\sqrt{R^2+z^2+2zL+L^2} - \sqrt{R^2+z^2-2zL+L^2}$$

$$= \frac{2L}{z} - \frac{LR^2}{z^3} + \cdots O\left(\frac{1}{z^4}\right) \qquad \text{if } z \gg L, R$$

$$\vec{E}_p(z) \approx \frac{\rho L \hat{z}}{\varepsilon_0} - \frac{\rho \hat{z} z}{2\varepsilon_0}\left(\frac{2L}{z} - \frac{LR^2}{z^3}\right)$$

$$= \frac{\rho L R^2}{2\varepsilon_0 z^2} = \underbrace{\frac{\rho}{4\pi\varepsilon_0 z^2}\left(2\pi R^2 L\right)}_{\text{Field of point charge}}$$

>

Symbolic Calculations for A1

> $integrate\left(\frac{r}{\left(z^2+r^2\right)^{\left(\frac{3}{2}\right)}},\ r=0..R\right)$ *assuming* $z>0$;

$$\frac{\sqrt{z^2+R^2}-z}{z\sqrt{z^2+R^2}} \quad (1)$$

> $integrate\left(\frac{r}{\left(z^2+r^2\right)^{\left(\frac{3}{2}\right)}},\ r\right)$;

$$-\frac{1}{\sqrt{z^2+r^2}} \quad (2)$$

> $integrate\left(\frac{(z-zp)}{\left(R^2+(z-zp)^2\right)^{\left(\frac{1}{2}\right)}},\ zp=-L..L\right)$;

$$\sqrt{L^2+2\,z\,L+z^2+R^2}-\sqrt{L^2-2\,z\,L+z^2+R^2} \quad (3)$$

> $f(z) := integrate\left(\frac{(z-zp)}{\left(R^2+(z-zp)^2\right)^{\left(\frac{1}{2}\right)}},\ zp=-L..L\right)$;

$$f := z \rightarrow integrate\left(\frac{z-zp}{\sqrt{R^2+(z-zp)^2}},\ zp=-L..L\right) \quad (4)$$

> $f(z)$;

$$\sqrt{L^2+2\,z\,L+z^2+R^2}-\sqrt{L^2-2\,z\,L+z^2+R^2} \quad (5)$$

> $f\left(\frac{1}{zz}\right)$;

$$-\frac{1}{\sqrt{\frac{R^2\,zz^2+1+2\,L\,zz+L^2\,zz^2}{zz^2}}\ zz^2\sqrt{\frac{R^2\,zz^2+1-2\,L\,zz+L^2\,zz^2}{zz^2}}}\Bigg(-\left(\frac{R^2\,zz^2+1-2\,L\,zz+L^2\,zz^2}{zz^2}\right)^{\frac{1}{2}} R^2\,zz^2$$
$$-\sqrt{\frac{R^2\,zz^2+1-2\,L\,zz+L^2\,zz^2}{zz^2}}-2\sqrt{\frac{R^2\,zz^2+1-2\,L\,zz+L^2\,zz^2}{zz^2}}\ L\,zz$$
$$-\sqrt{\frac{R^2\,zz^2+1-2\,L\,zz+L^2\,zz^2}{zz^2}}\ L^2\,zz^2+\sqrt{\frac{R^2\,zz^2+1+2\,L\,zz+L^2\,zz^2}{zz^2}}\ R^2\,zz^2$$
$$+\sqrt{\frac{R^2\,zz^2+1+2\,L\,zz+L^2\,zz^2}{zz^2}}-2\sqrt{\frac{R^2\,zz^2+1+2\,L\,zz+L^2\,zz^2}{zz^2}}\ L\,zz \quad (6)$$

$$+\sqrt{\frac{R^2 zz^2+1+2\,L\,zz+L^2\,zz^2}{zz^2}}\;L^2\,zz^2\Big)$$

$$>\ taylor\left(f\left(\frac{1}{zz}\right),\ zz=0\right);$$

$$2\,L+\Big(-5\,L^3+4\,L\,\Big(-\frac{1}{2}\,R^2+L^2\Big)+\frac{1}{2}\,L\,\big(R^2+L^2\big)+\Big(\frac{1}{2}\,R^2+\frac{1}{2}\,L^2\Big)\,L\Big)\,zz^2 \tag{7}$$

$$+\Big(-\Big(-2\,L\,\Big(-\frac{1}{2}\,R^2+L^2\Big)+3\,L^3-\frac{1}{2}\,L\,\big(R^2+L^2\big)-\Big(\frac{1}{2}\,R^2+\frac{1}{2}\,L^2\Big)\,L\Big)\,L$$

$$+2\,L\,\Big(-\frac{3}{4}\,L\,\big(R^2+L^2\big)+\frac{3}{2}\,\Big(-\frac{1}{2}\,R^2-\frac{1}{2}\,L^2\Big)\,L+\frac{5}{2}\,L^3\Big)+2\,L\,\Big(\frac{3}{4}\,L\,\big(R^2$$

$$+L^2\big)-\frac{3}{2}\,\Big(-\frac{1}{2}\,R^2-\frac{1}{2}\,L^2\Big)\,L-\frac{5}{2}\,L^3\Big)+\Big(3\,L^3-\frac{1}{2}\,L\,\big(R^2+L^2\big)-\Big(\frac{1}{2}\,R^2+\frac{1}{2}\,L^2\Big)\,L\Big)\,L$$

$$+2\,\Big(\frac{1}{4}\,L\,\big(R^2+L^2\big)+\frac{1}{2}\,\Big(\frac{1}{2}\,R^2+\frac{1}{2}\,L^2\Big)\,L-\frac{1}{2}\,L^3\Big)\,L+2\,\Big(-\frac{1}{4}\,L\,\big(R^2+L^2\big)-\frac{1}{2}\,\Big(\frac{1}{2}\,R^2$$

$$+\frac{1}{2}\,L^2\Big)\,L+\frac{1}{2}\,L^3\Big)\,L+\Big(-\frac{1}{4}\,L\,\big(R^2+L^2\big)-\frac{1}{2}\,\Big(\frac{1}{2}\,R^2+\frac{1}{2}\,L^2\Big)\,L\Big)\,L+\Big(\frac{1}{4}\,L\,\big(R^2+L^2\big)$$

$$+\frac{1}{2}\,\Big(\frac{1}{2}\,R^2+\frac{1}{2}\,L^2\Big)\,L\Big)\,L-2\,L^2\,\Big(-\frac{1}{2}\,R^2+L^2\Big)\Big)\,zz^3+\Big(-\Big(-2\,L\,\Big(\frac{3}{4}\,L\,\big(R^2$$

$$+L^2\big)-\frac{3}{2}\,\Big(-\frac{1}{2}\,R^2-\frac{1}{2}\,L^2\Big)\,L-\frac{5}{2}\,L^3\Big)-\Big(3\,L^3-\frac{1}{2}\,L\,\big(R^2+L^2\big)-\Big(\frac{1}{2}\,R^2$$

$$+\frac{1}{2}\,L^2\Big)\,L\Big)\,L-2\,\Big(\frac{1}{4}\,L\,\big(R^2+L^2\big)+\frac{1}{2}\,\Big(\frac{1}{2}\,R^2+\frac{1}{2}\,L^2\Big)\,L-\frac{1}{2}\,L^3\Big)\,L-2\,\Big(-\frac{1}{4}\,L\,\big(R^2$$

$$+L^2\big)-\frac{1}{2}\,\Big(\frac{1}{2}\,R^2+\frac{1}{2}\,L^2\Big)\,L+\frac{1}{2}\,L^3\Big)\,L-\Big(-\frac{1}{4}\,L\,\big(R^2+L^2\big)-\frac{1}{2}\,\Big(\frac{1}{2}\,R^2$$

$$+\frac{1}{2}\,L^2\Big)\,L\Big)\,L-\Big(\frac{1}{4}\,L\,\big(R^2+L^2\big)+\frac{1}{2}\,\Big(\frac{1}{2}\,R^2+\frac{1}{2}\,L^2\Big)\,L\Big)\,L\Big)\,L-\Big(-2\,L\,\Big(-\frac{1}{2}\,R^2+L^2\Big)$$

$$+3\,L^3-\frac{1}{2}\,L\,\big(R^2+L^2\big)-\Big(\frac{1}{2}\,R^2+\frac{1}{2}\,L^2\Big)\,L\Big)\,\Big(-\frac{1}{2}\,R^2+L^2\Big)-2\,L^2\,\Big(-\frac{3}{4}\,L\,\big(R^2+L^2\big)$$

$$+\frac{3}{2}\,\Big(-\frac{1}{2}\,R^2-\frac{1}{2}\,L^2\Big)\,L+\frac{5}{2}\,L^3\Big)+2\,L\,\Big(-\frac{3}{4}\,\Big(-\frac{1}{2}\,R^2-\frac{1}{2}\,L^2\Big)\,\big(R^2+L^2\big)-\frac{5}{4}\,L^2\,\big(R^2$$

$$+L^2\big)+\frac{5}{3}\,\Big(-\frac{3}{4}\,L\,\big(R^2+L^2\big)+\frac{3}{2}\,\Big(-\frac{1}{2}\,R^2-\frac{1}{2}\,L^2\Big)\,L\Big)\,L+\frac{35}{8}\,L^4\Big)$$

$$+2\,L\,\Big(-\frac{3}{4}\,\Big(-\frac{1}{2}\,R^2-\frac{1}{2}\,L^2\Big)\,\big(R^2+L^2\big)-\frac{5}{4}\,L^2\,\big(R^2+L^2\big)-\frac{5}{3}\,\Big(\frac{3}{4}\,L\,\big(R^2$$

$$+L^2\big)-\frac{3}{2}\,\Big(-\frac{1}{2}\,R^2-\frac{1}{2}\,L^2\Big)\,L\Big)\,L+\frac{35}{8}\,L^4\Big)+2\,\Big(-\frac{1}{4}\,\Big(\frac{1}{2}\,R^2+\frac{1}{2}\,L^2\Big)\,\big(R^2+L^2\big)$$

$$+\frac{1}{4}\,L^2\,\big(R^2+L^2\big)-\Big(-\frac{1}{4}\,L\,\big(R^2+L^2\big)-\frac{1}{2}\,\Big(\frac{1}{2}\,R^2+\frac{1}{2}\,L^2\Big)\,L\Big)\,L-\frac{5}{8}\,L^4\Big)\,L$$

$$+2\,\Big(-\frac{1}{4}\,\Big(\frac{1}{2}\,R^2+\frac{1}{2}\,L^2\Big)\,\big(R^2+L^2\big)+\frac{1}{4}\,L^2\,\big(R^2+L^2\big)+\Big(\frac{1}{4}\,L\,\big(R^2+L^2\big)+\frac{1}{2}\,\Big(\frac{1}{2}\,R^2$$

$$+\frac{1}{2}\,L^2\Big)\,L\Big)\,L-\frac{5}{8}\,L^4\Big)\,L+\Big(\frac{1}{4}\,L\,\big(R^2+L^2\big)+\frac{1}{2}\,\Big(\frac{1}{2}\,R^2$$

$$+\frac{1}{2}\,L^2\Big)\,L-\frac{1}{2}\,L^3\Big)\,R^2-\Big(-\frac{1}{4}\,L\,\big(R^2+L^2\big)-\frac{1}{2}\,\Big(\frac{1}{2}\,R^2+\frac{1}{2}\,L^2\Big)\,L$$

$$+\frac{1}{2}L^3\Big)L^2-\Big(-\frac{1}{4}L\left(R^2+L^2\right)-\frac{1}{2}\left(\frac{1}{2}R^2+\frac{1}{2}L^2\right)L+\frac{1}{2}L^3\Big)R^2+\frac{1}{2}\Big(-\frac{1}{4}L\big(R^2$$

$$+L^2\big)-\frac{1}{2}\left(\frac{1}{2}R^2+\frac{1}{2}L^2\right)L\Big)\left(R^2+L^2\right)-\frac{1}{2}\left(\frac{1}{2}R^2+\frac{1}{2}L^2\right)\left(R^2+L^2\right)L+\frac{5}{8}L^3\big(R^2$$

$$+L^2\big)+\frac{5}{4}\Big(\frac{1}{4}L^2\left(R^2+L^2\right)-\Big(-\frac{1}{4}L\left(R^2+L^2\right)-\frac{1}{2}\Big(\frac{1}{2}R^2$$

$$+\frac{1}{2}L^2\Big)L\Big)L\Big)L-\frac{7}{4}L^5-\frac{1}{2}\left(\frac{1}{4}L\left(R^2+L^2\right)+\frac{1}{2}\left(\frac{1}{2}R^2+\frac{1}{2}L^2\right)L\right)\left(R^2+L^2\right)$$

$$+\frac{5}{4}\left(\frac{1}{4}L^2\left(R^2+L^2\right)+\left(\frac{1}{4}L\left(R^2+L^2\right)+\frac{1}{2}\left(\frac{1}{2}R^2+\frac{1}{2}L^2\right)L\right)L\right)L+\Big(\frac{1}{4}L\left(R^2+L^2\right)$$

$$+\frac{1}{2}\left(\frac{1}{2}R^2+\frac{1}{2}L^2\right)L-\frac{1}{2}L^3\Big)L^2-\left(3L^3-\frac{1}{2}L\left(R^2+L^2\right)-\left(\frac{1}{2}R^2+\frac{1}{2}L^2\right)L\right)\Big(-\frac{1}{2}R^2$$

$$+L^2\Big)+\Big(-2\left(\frac{1}{4}L\left(R^2+L^2\right)+\frac{1}{2}\left(\frac{1}{2}R^2+\frac{1}{2}L^2\right)L-\frac{1}{2}L^3\right)L-2\Big(-\frac{1}{4}L\big(R^2$$

$$+L^2\big)-\frac{1}{2}\left(\frac{1}{2}R^2+\frac{1}{2}L^2\right)L+\frac{1}{2}L^3\Big)L-\Big(-\frac{1}{4}L\left(R^2+L^2\right)-\frac{1}{2}\Big(\frac{1}{2}R^2$$

$$+\frac{1}{2}L^2\Big)L\Big)L-\left(\frac{1}{4}L\left(R^2+L^2\right)+\frac{1}{2}\left(\frac{1}{2}R^2+\frac{1}{2}L^2\right)L\right)L\Big)L\Big)zz^4+\mathrm{O}\left(zz^5\right)$$

> *simplify*(%);

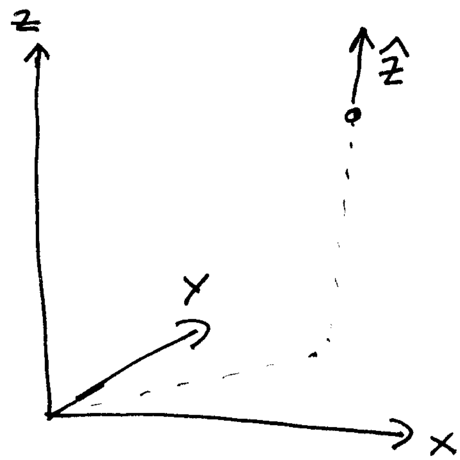
$$2L-LR^2zz^2+\left(-R^2L^3+\frac{3}{4}LR^4\right)zz^4+\mathrm{O}\left(zz^5\right) \tag{8}$$

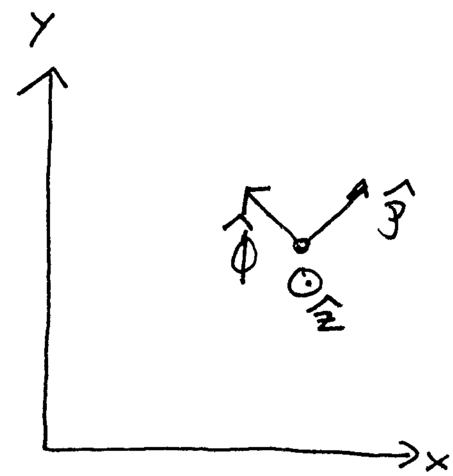
>

AZ

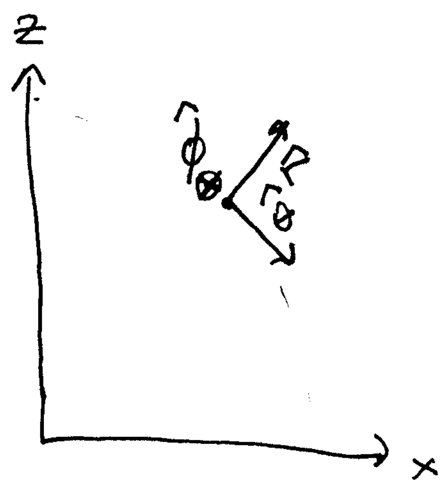
(a)

Cylindrical





Spherical



(b) Cylindrical $\qquad \hat{z} = \hat{z}$

$$\hat{\rho} = \left(\frac{x}{\sqrt{x^2+y^2}} , \frac{y}{\sqrt{x^2+y^2}} , 0 \right)$$

$$\hat{\phi} = \left(\frac{-y}{\sqrt{x^2+y^2}} , \frac{x}{\sqrt{x^2+y^2}} , 0 \right)$$

Spherical $\hat{\phi}$ same

$$\hat{r} = \frac{\vec{r}}{r} = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right)$$

To build $\hat{\theta}$ observe from our drawing that $\hat{\phi} \times \hat{r} = \hat{\theta}$

$$\hat{\theta} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\frac{y}{\rho} & \frac{x}{\rho} & 0 \\ \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \end{vmatrix}$$

$$= \frac{xz}{\rho r} \hat{x} + \frac{yz}{\rho r} \hat{y} - \frac{x^2+y^2}{\rho r} \hat{z}$$

Wolfram Claims

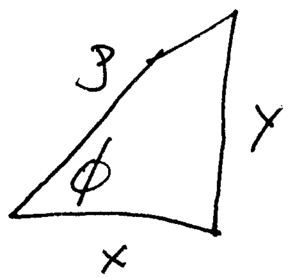
$$\hat{\theta} = (\cos\theta \cos\phi, \cos\theta \sin\phi, -\sin\theta)$$

How do we make sense of this?

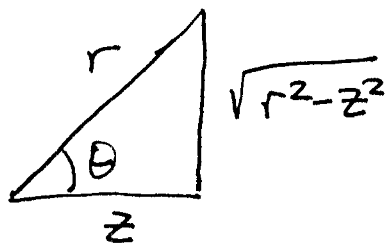
From the geometry

$$\hat{r}\cdot\hat{z} = \cos\theta = \frac{z}{r}$$

$$\hat{\rho}\cdot\hat{x} = \cos\phi = \frac{x}{\rho}$$



$$\Longrightarrow \quad \sin\phi = \frac{y}{\rho}$$



$$\sin\theta = \frac{\sqrt{r^2-z^2}}{r} = \frac{\sqrt{x^2+y^2}}{r}$$

$$= \frac{x^2+y^2}{\rho r}$$

So everything is perfect

How would we do this from the identites ?

$$x = r\cos\phi\sin\theta$$

$$y = r\sin\phi\sin\theta$$

$$z = r\cos\theta$$

$$x^2+y^2 = r^2\sin^2\theta\left(\cos^2\phi + \sin^2\phi\right)$$

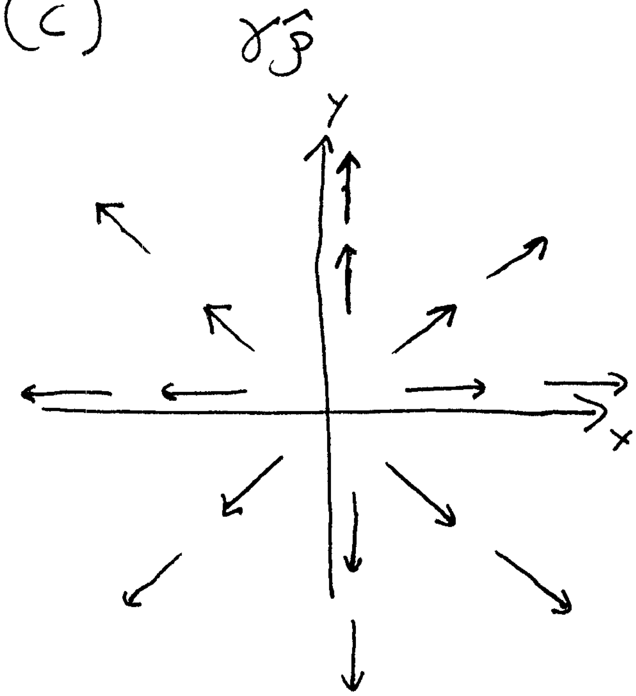
$$= r^2\sin^2\theta$$

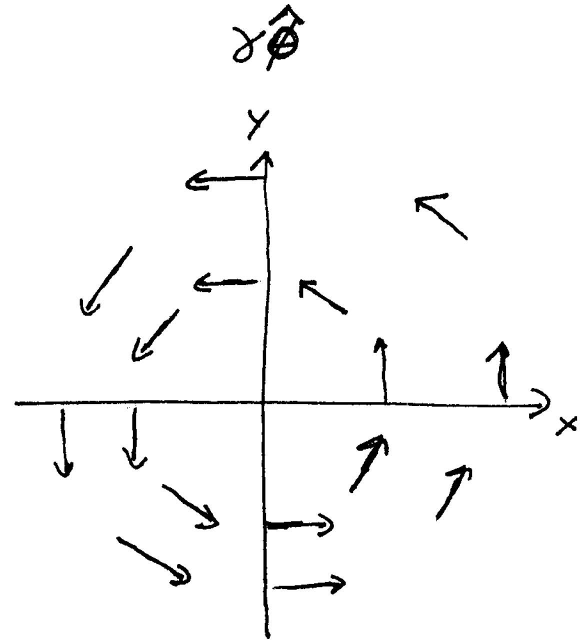
$$\sin\theta = \frac{\sqrt{x^2+y^2}}{r} \qquad \cos\theta = \frac{z}{r}$$

$$x = r\cos\phi\,\frac{\sqrt{x^2+y^2}}{r}$$

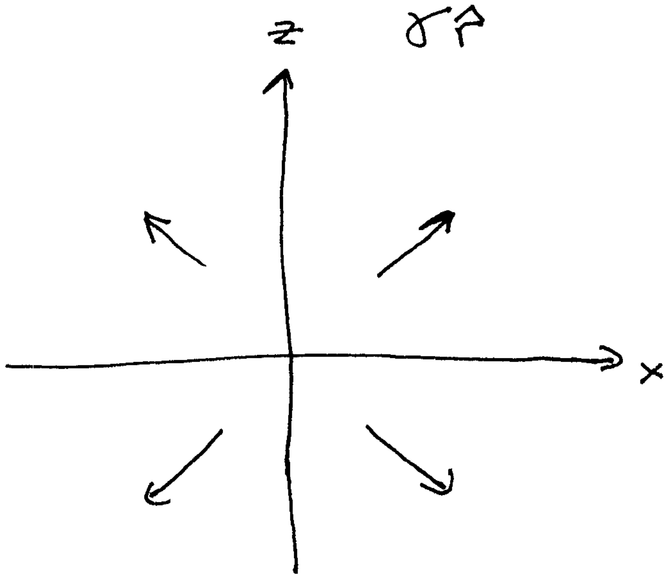
$$\cos\phi = \frac{x}{\sqrt{x^2+y^2}} \qquad \sin\phi = \frac{y}{\sqrt{x^2+y^2}}$$

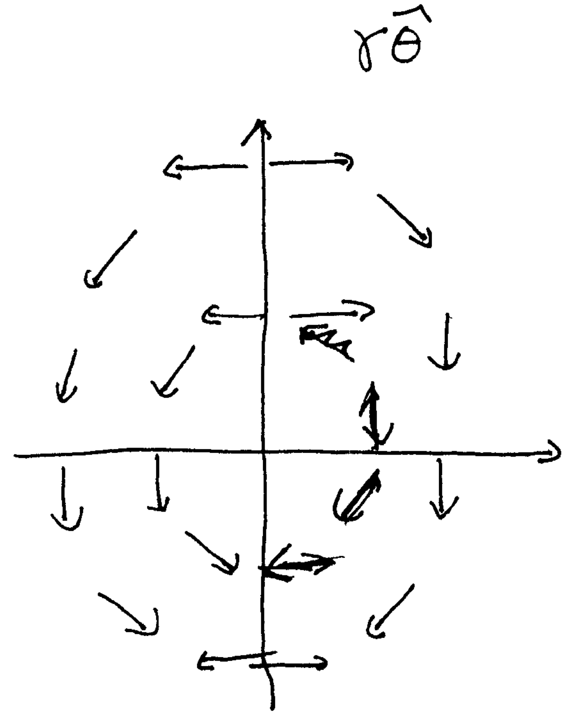
(c)





Vectors parallel to x-y plane





Vectors Point Out from Origin

(d) For $r\hat{\rho}$ and $r\hat{r}$ $\nabla \cdot \vec{E} \neq 0$ because there is net flow outward, but $\nabla \times \vec{E} = 0$, no circulation.

For $r\hat{\theta}$ and $r\hat{\phi}$ $\nabla \times \vec{E} \neq 0$ (circulation) but $\nabla \cdot \vec{E} = 0$ no net flow for $\hat{\phi}$, $\nabla \cdot \vec{E} \neq 0$ at $\theta = 0$ for $\hat{\theta}$

(e)

$$\nabla \cdot \hat{\rho} = \frac{1}{\rho}\frac{\partial \rho}{\partial \rho} = \frac{1}{\rho}$$ cylindrical

$$\nabla \times \hat{\rho} = 0$$

$$\nabla \cdot \hat{r} = \frac{1}{r^2}\frac{\partial r^2}{\partial r} = \frac{2}{r}$$ spherical

$$\nabla \times \hat{r} = 0$$

$$\nabla \times \hat{\phi} = \frac{1}{\rho}\frac{\partial \rho}{\partial \rho}\hat{z} = \frac{1}{\rho}\hat{z}$$ cylindrical

$$\nabla \cdot \hat{\phi} = 0$$

$$\nabla \times \hat{\theta} = \frac{1}{r}\frac{\partial r}{\partial r}\hat{\phi} = \frac{1}{r}\hat{\phi}$$ spherical

$$\nabla \cdot \hat{\theta} = \frac{1}{r\sin\theta}\frac{\partial \sin\theta}{\partial \theta} = \frac{\cot\theta}{r}$$

(f) For $\hat{r}$

$$\nabla \cdot \hat{r} = \frac{\partial}{\partial x}\frac{x}{r} + \frac{\partial}{\partial y}\frac{y}{r} + \frac{\partial}{\partial z}\frac{z}{r}$$

$$= \frac{3}{r} + \left(\frac{-x^2}{r^3} + \frac{-y^2}{r^3} + \frac{-z^2}{r^3}\right)$$

$$= \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

$$\nabla \times \hat{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \end{vmatrix}$$

$$= \hat{x}\left(-\frac{yz}{r^3} + \frac{yz}{r^3}\right) - \hat{y}\left(-\frac{xz}{r^3} + \frac{xz}{r^3}\right)$$

$$+ \hat{z}\left(\frac{-xy}{r^3} + \frac{xy}{r^3}\right) = 0$$

A2-9

$$\nabla \cdot \hat{\phi} = \frac{\partial}{\partial x}\left(\frac{-y}{\rho}\right) + \frac{\partial}{\partial y}\left(\frac{x}{\rho}\right)$$

$$= \frac{xy}{\rho^3} - \frac{xy}{\rho^3} = 0$$

$$\nabla \times \hat{\phi} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{\rho} & \frac{x}{\rho} & 0 \end{vmatrix}$$

$$= 0\hat{x} - 0\hat{y} + \hat{z}\underbrace{\left(\frac{1}{\rho} - \frac{x^2}{\rho^3} + \frac{1}{\rho} - \frac{y^2}{\rho^3}\right)}_{\frac{2}{\rho} - \frac{\rho^2}{\rho^3}}$$

$$= \frac{\hat{z}}{\rho}$$

A3 (a) $Q = \frac{4}{3}\pi R^3 \rho$

$$= \frac{4}{3}\pi\left(100\times10^{-6}\frac{C}{m^3}\right)\left(5\times10^{-2}m\right)^3$$

$$= 5.2\times10^{-8}C$$

(b) $Q = 2\pi R L \sigma$

$$= 2\pi(0.06m)(0.2m)(10\times10^{-6}C/m^2)$$

$$= 7.5\times10^{-7}C$$

A4 The electric force is vastly stronger than the gravitational force.

A5

$$F = \frac{kq_1q_2}{d^2} = \frac{\left(9\times10^9\frac{Nm^2}{C^2}\right)\left(2\times10^6 C\right)^2}{(0.1m)^2}$$

$$= 36\times10^{23}N = 3.6\times10^{24}N$$