PHYS 3414 - Electricity and Magnetism- Homework Set 10

Due Wednesday April 30th at 5:00pm

Good Problems

12.6

12.8 ($v=0$) 20pts

12.16 20pts

12.10 (Just find force) 20pts

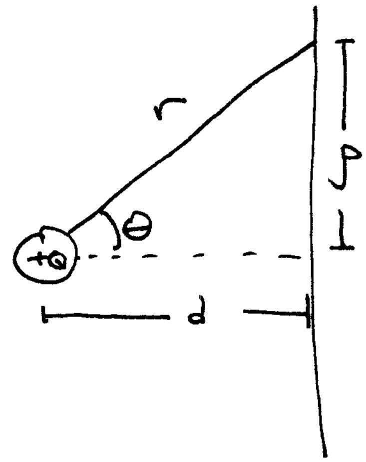
12.22 20 pts

12.24

12.26 Report integral.

12.28 20pts

(12.6) The bound charge density is given in eqn 12.13



$$\sigma_b = -\frac{(\varepsilon_r - 1)\, Q \cos\theta}{2\pi(\varepsilon_r + 1)\, r^2}$$

$\rho = r\sin\theta$ $\qquad d = r\cos\theta$

$r^2 = d^2 + \rho^2$

$$\sigma_b = \frac{-(\varepsilon_r - 1)\, Q}{2\pi(\varepsilon_r + 1)} \frac{d}{r^3}$$

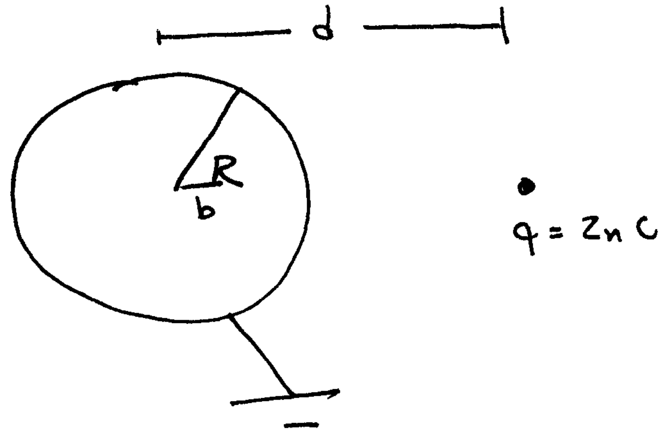
$$= \frac{-(\varepsilon_r - 1)\, Q}{2\pi(\varepsilon_r + 1)} \frac{d}{\left(\sqrt{d^2 + \rho^2}\right)^3}$$

$$Q = \int_0^\infty 2\pi\rho\, d\rho\, \sigma_b = \frac{-(\varepsilon_r - 1)\, Q\, d\, 2\pi}{2\pi(\varepsilon_r + 1)} \int_0^\infty \frac{\rho\, d\rho}{(d^2 + \rho^2)^{3/2}}$$

$$= \frac{(\varepsilon_r - 1)\, Q}{\varepsilon_r + 1} \left[\frac{d}{(d^2 + \rho^2)^{1/2}}\right]_0^\infty$$

$$= -\frac{(\varepsilon_r - 1)}{(\varepsilon_r + 1)} Q$$

12.8



Note, because grounded conducting sphere will have a net charge. The charge will be such that it brings surface of sphere to V=0

Image charge to bring surface to V = 0

(12.16) $$Q' = -\frac{R}{d} Q$$

$$= -\frac{7cm}{20cm} 2nC = -\frac{7}{10} nC$$

(12.17) The image charge must be placed a distance

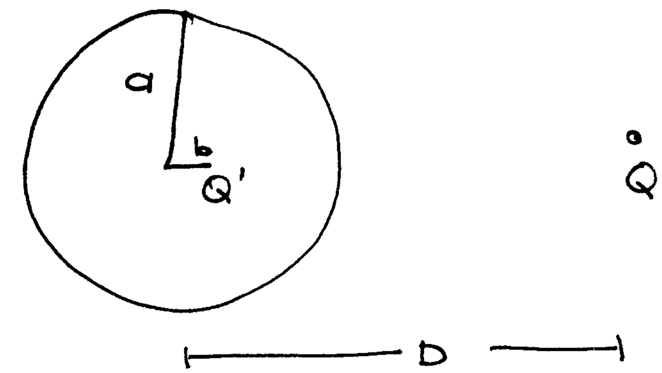
$$b = \frac{R^2}{d} = \frac{(0.07m)^2}{(0.20m)} = 0.0245m$$

The distance between the real charge and the image charge is $l = d-b$ and the force is

$$F = \frac{kQQ'}{(d-b)^2} = \frac{\left(9\times10^{9}\,\frac{Nm^2}{C^2}\right)\left(2\times10^{-9}C\right)\left(-\frac{7}{10}\times10^{-9}C\right)}{(0.2m - 0.0245m)^2}$$

$$= 4.1\times10^{-7}N \quad \text{attractive}$$

(12.10)



Let Q_s be the total charge of the sphere. If the sphere were at zero potential an image charge

$$Q' = -\frac{a}{D} Q \qquad \text{at} \qquad b = \frac{a^2}{D}$$

would be needed.

With the image charge, the net charge of the sphere is free to uniformly spread out over the surface.

Note this meets all boundary conditions

(1) Total charge $= Q_s$

(2) $\vec{E} \parallel \hat{n}$

Now how much charge do we need? The image charge charge is physically spread over the surface so

$$Q_s = Q - Q'$$

The force on Q is

$$F = \frac{KQQ_s}{D^2} + \frac{KQQ'}{(D-b)^2}$$

Note $Q' < 0$

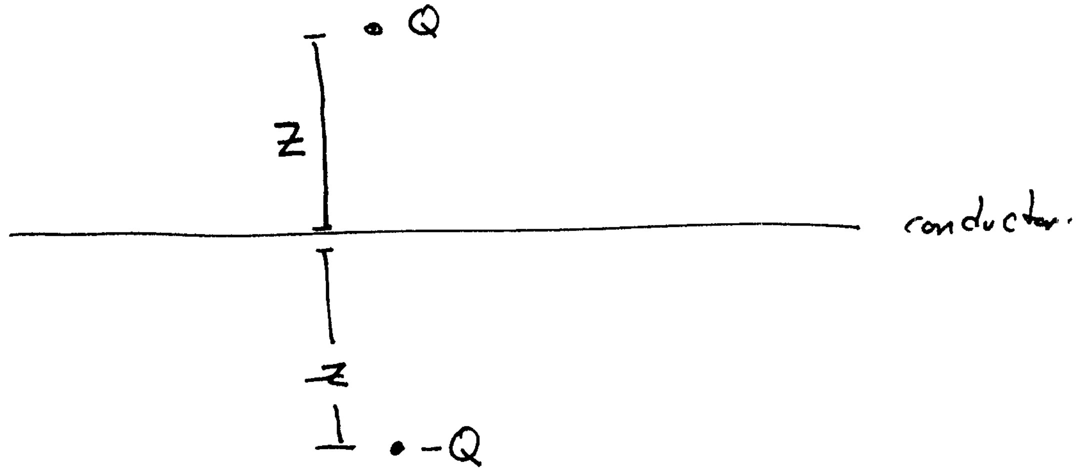
$$= \frac{K(Q-Q')}{D^2} + \frac{KQQ'}{(D-b)^2}$$

This force is zero at

$$-(Q-Q')(D-b)^2 = QQ'D^2$$

which has a rather complicated solution.

12.16



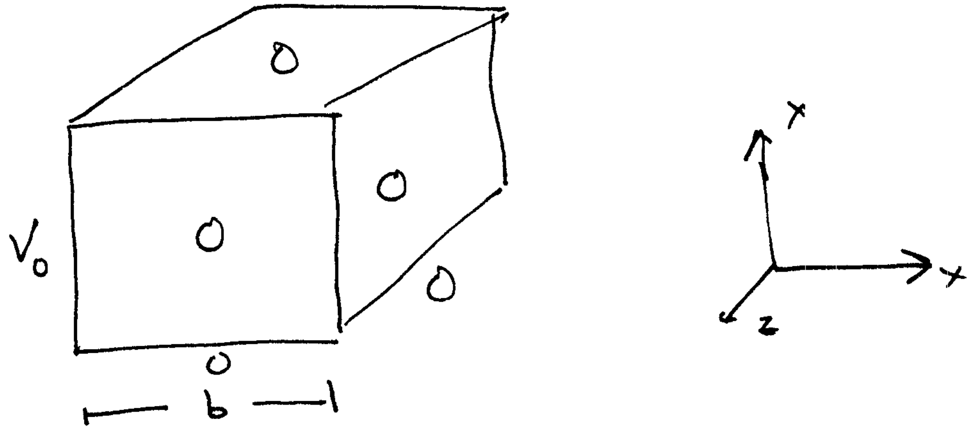
$$F = \frac{kQ^2}{(2z)^2}$$

$$W = \int_D^\infty F\,dz = \frac{kQ^2}{4}\int_D^\infty \frac{dz}{z^2}$$

$$= -\frac{kQ^2}{4}\left[\frac{1}{z}\right]_D^\infty = \frac{kQ^2}{4D}$$

* Note, work positive because some external agent is applying the force.

12.22



Separate the Laplacian

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

$$w^2 \quad -k^2 \quad - l^2 = 0$$

$$X'' - w^2 X = 0 \quad \Rightarrow \quad X = e^{\pm wx}$$

$$Y'' + k^2 Y = 0 \quad \Rightarrow \quad Y = \sin ky, \cos ky$$

$$Z'' + l^2 Z = 0 \quad \Rightarrow \quad Z = \sin lz, \cos lz$$

Apply y, z boundary conditions

$$Y(0) = 0 \qquad Y(b) = 0 \quad \Rightarrow \quad k = \frac{m\pi}{b}$$

$$Z(0) = 0 \qquad Z(b) = 0 \qquad l = \frac{n\pi}{b}$$

for both cases choose sine solution.

$$W_{mn}^2 = k^2 + l^2 = \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \left(\frac{\pi}{b}\right)^2 (m^2+n^2)$$

$$V(x,y,z) = \sum_{m,n} \left(A_{nm} e^{W_{nm} x} + B_{nm} e^{-W_{nm} x}\right) \sin k_m y \sin l_n z$$

Apply $V(b,y,z) = 0$ boundary condition. This would be automatically satisfied by a combination of exponentials $\sinh\left(1-\frac{x}{b}\right) b W_{nm}$

Note, for more brute force solution see second method of solution.

Let $\omega_{nm} = b W_{nm} = \pi\sqrt{n^2+m^2}$

$$V(x,y,z) = \sum_{n,m} A'_{nm} \sinh \omega_{nm} \left(1-\frac{x}{b}\right) \sin k_m y \sin l_m z$$

Apply final boundary condition $V(0, x, z) = V_0$

$$V(0, y, z) = \sum_{n,m} A'_{nm} \sinh \omega_{nm} \sin k_m y \sin \ell_n z$$

$$= V_0$$

Use orthogonality (twice)

$$\int_0^b \sin k_m y \sin k_{m'} y \, dy = \frac{b}{2} \quad \text{if } m = m'$$

$$= 0 \quad \text{otherwise}$$

Multiply both sides by $\sin k_{m'} y \sin \ell_{n'} z$ and integrate

$$V_0 \int_0^b dy \int_0^b dz \, \sin k_{m'} y \sin k_{n'} z$$

$$= \sum_{n,m} A'_{nm} \sinh(\omega_{nm}) \int_0^b \cancel{\sin} \int_0^b \sin k_{m'} y \sin k_m y \, \sin k_{n'} z \sin k_n z \, dy \, dz$$

$$= \left(\frac{b}{2}\right)^2 A'_{n'm'} \sinh(\omega_{nm})$$

$$A'_{n'm'} = \frac{4V_0}{b^2 \sinh \omega_{nm}} \int_0^b \int_0^b \sin k_{m'} y \sin k_{n'} z \, dy \, dz$$

If n', m' even, $A'_{n'm'} = 0$

If n' and m' odd,

$$\int_0^b \sin \frac{m\pi}{b} y \, dy = \frac{2b}{m\pi}$$

$$\int_0^b \sin \frac{n\pi}{b} z \, dz = \frac{2b}{n\pi}$$

$$A'_{n'm'} = \frac{16 V_0}{\pi^2 nm \sinh \omega_{nm}} = \frac{16 V_0}{\pi^2 nm \sinh(\pi\sqrt{n^2+m^2})}$$

Final Result

$$V = \sum_{n,m\,odd} \left(\frac{16 V_0}{\pi^2 nm}\right) \frac{\sinh\left(\pi\sqrt{n^2+m^2}\left(1-\frac{x}{b}\right)\right)}{\sinh(\pi\sqrt{n^2+m^2})} \sin\frac{m\pi}{b}y \sin\frac{n\pi}{b}z$$

Now for the brute force method not involving sinh.

Brute Force

$$w_{nm}^2 = \frac{\pi^2}{b^2}(n^2+m^2)$$

$$V(x,y,z) = \sum_{m,n}\left(A_{nm}e^{w_{nm}x} + B_{nm}e^{-w_{nm}x}\right)\sin k_m y \, \sin l_n z$$

Apply $V(b,y,z)=0$ boundary condition.

$$V(b,y,z) = \sum_{m,n}\left(A_{nm}e^{w_{nm}b} + B_{nm}e^{-w_{nm}b}\right)\sin k_m y \, \sin l_n z$$

Since $\sin k_m y$ $\sin l_n z$ are independent each term is ( ) must be zero.

$$A_{nm}e^{w_{nm}b} + B_{nm}e^{-w_{nm}b} = 0$$

$$A_{nm} = -B_{nm}e^{-2w_{nm}b}$$

$$V(x,y,z) = \sum_{m,n} B_{mn}\left(e^{-w_{nm}x} - e^{w_{nm}x - 2w_{nm}b}\right)\sin k_m y \, \sin l_n z$$

Apply $V(0,y,z) = \cancel{V_0}$ boundary condition

$$V(0,y,z) = V_0 = \sum_{n,m} B_{mn}\left(1 - e^{-2w_{nm}b}\right) \sin k_m y \sin l_n z$$

Use orthogonality again as in previous solution

$$V_0 \int_0^b \sin k_{m'}y \sin l_{n'}z \, dy \, dz$$

$$= \left(\frac{b}{2}\right)^2 B_{m'n'}\left(1 - e^{-2w_{n'm'}b}\right)$$

$$B_{m'n'} = \frac{4V_0}{b^2\left(1 - e^{-2w_{n'm'}b}\right)} \int_0^b \int_0^b \sin k_{m'}y \sin k_{n'}z \, dy \, dy$$

If m', n' even, $= 0$.

If m', n' odd

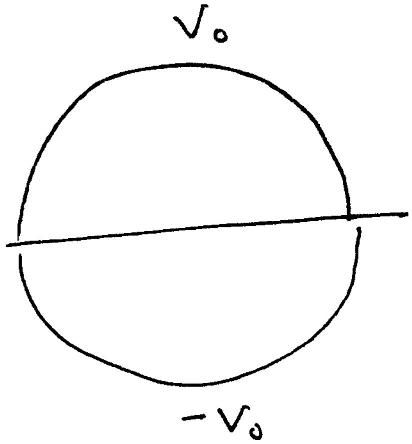
$$B_{m'n'} = \frac{16 V_0}{\pi^2 nm\left(1 - e^{-2w_{n'm'}b}\right)}$$

Finally

$$V(x,y,z) = \sum_{m,n} \frac{16V_0\left(e^{-w_{nm}x} - e^{+w_{nm}x-2w_{nm}b}\right)}{\pi^2 nm\left(1-e^{-2w_{nm}b}\right)}$$

$$\times \sin k_m y \ \sin q_n z$$

12.24



Inside

$$V(r,\theta) = \sum_n A_n r^n P_n(\cos\theta)$$

Completeness

$$\int_0^{\pi} + \sin\theta \, d\theta \, P_n(\cos\theta) \, P_m(\cos\theta)$$

$$= \frac{2}{2n+1} \delta_{mn} \qquad (12.36)$$

$$= \int_{-1}^{1} d(\cos\theta) \, P_n(\cos\theta) \, P_m(\cos\theta)$$

$$\int_{-1}^{1} V(\theta)\, P_m(\cos\theta)\, d(\cos\theta)$$

$$= \sum_n A_n a^n \int_{-1}^{1} P_m(\cos\theta)\, P_n(\cos\theta)\, d(\cos\theta)$$

$$= \frac{2A_m a^m}{2m+1}$$

$$A_n = \frac{2n+1}{2a^n} \int_{-1}^{1} V(\theta)\, P_n(\cos\theta)\, d(\cos\theta)$$

$x = \cos\theta$

$$= \frac{2n+1}{2a^n} \int_{-1}^{1} V(x)\, P_n(x)\, dx$$

$P_0 = 1$ $\quad P_1 = x$ $\quad P_2 = \frac{1}{2}(x^2 - 1)$

$V(x) = -V_0 \quad x \in (-1, 0)$

$V(x) = V_0 \quad x \in (0, 1)$

$A_0 = 0$, $A_2 = 0$ even function, odd range

$$A_1 = \frac{2+1}{2a}\left[\int_{-1}^{0} -V_0 x\,dx + \int_{0}^{1} V_0 x\,dx\right]$$

$$\frac{2V_0}{2}$$

$$= \frac{3V_0}{2a}$$

Outside

$$V(r,\theta) = \sum B_n r^{-(n+1)} P_n(\cos\theta)$$

Same logic

$$B_n = \frac{2n+1}{2a^{-(n+1)}} \int_{-1}^{1} V(x) P_n(x)\,dx$$

$B_0 = 0$ $B_2 = 0$ $B_1 = \frac{2+1}{2a^{-2}} \cdot V_0 = \frac{3V_0 a^2}{2}$

12.26

Inside

$$V_i(r,\theta) = \sum A_n r^n P_n(\cos\theta)$$

Outside

$$V_o(r,\theta) = \sum B_n r^{-(n+1)} P_n(\cos\theta)$$

The potential must be continuous at $r=a$

$$V_i(a,\theta) = \sum A_n a^n P_n(\cos\theta)$$

$$= V_o(a,\theta) = \sum B_n a^{-(n+1)} P_n(\cos\theta)$$

Since P_n orthogonal, must be equal term by term

$$A_n a^n = B_n a^{-(n+1)}$$

Gauss' Law must also be satisfied at the surface

$$\vec{E}_o \cdot \hat{n} A - \vec{E}_i \cdot \hat{n} A = \frac{Q_{enc}}{\varepsilon_o} = \frac{\sigma(\theta) A}{\varepsilon_o}$$

$$\left.\frac{\partial V_o}{\partial r}\right|_a - \left.\frac{\partial V_i}{\partial r}\right|_a = -\frac{\sigma(\theta)}{\varepsilon_o}$$

$$\left.\frac{\partial V_o}{\partial r}\right|_a = \sum_n -(n+1)\, a^{-(n+2)} B_n P_n(\cos\theta)$$

$$\left.\frac{\partial V_i}{\partial r}\right|_a = \sum_n n\, a^{n-1} A_n P_n(\cos\theta)$$

$$\sigma(\theta) = \varepsilon_o \left(\left.\frac{\partial V_i}{\partial r}\right|_a - \left.\frac{\partial V_o}{\partial r}\right|_a \right)$$

$$= \varepsilon_o \sum_n n a^{n-1} A_n P_n(\cos\theta) + (n+1)\, a^{-(n+2)} B_n P_n(\cos\theta)$$

Using $A_n a^n = B_n a^{-(n+1)}$ we get

$$\sigma(\theta) = \varepsilon_0 \sum_n (2n+1) A_n a^{n-1} P_n(\cos\theta)$$

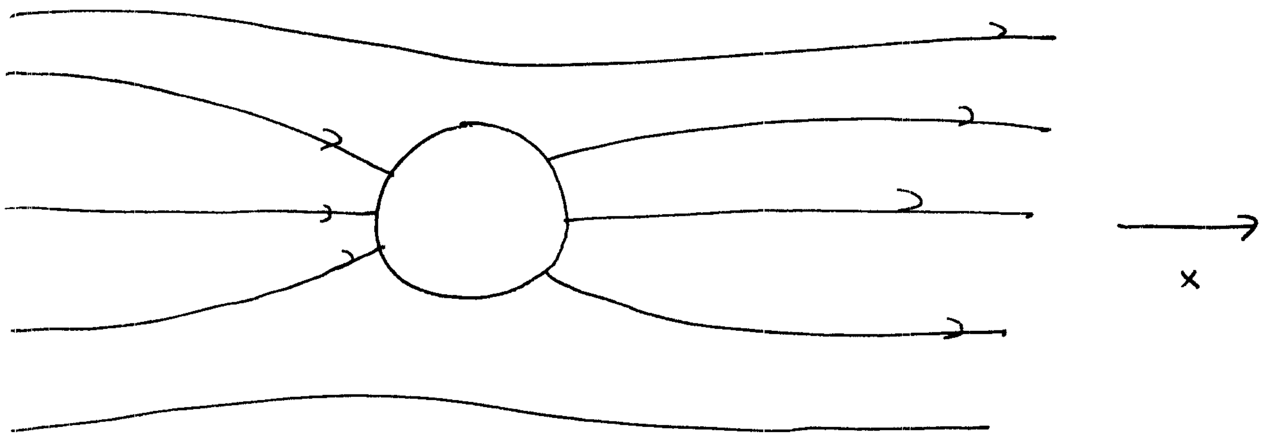
$$\int_{-1}^{1} d(\cos\theta)\, P_m(\cos\theta)\, \sigma(\theta) = \varepsilon_0 \sum_n (2n+1) A_n a^{n-1} \int_{-1}^{1} d(\cos\theta)\, P_n P_m$$

$$\int_{-1}^{1} d(\cos\theta)\, P_n P_m = \frac{2}{2n+1} \delta_{nm}$$

$$\int_{-1}^{1} d(\cos\theta)\, P_m(\cos\theta)\, \sigma(\theta) = \varepsilon_0 a^{n-1} 2 A_n$$

$$A_n = \frac{1}{2\varepsilon_0 a^{n-1}} \int_{-1}^{1} \sigma(\theta) P_m(\cos\theta)\, d\cos\theta$$

12.28



At long range, $\vec{E} = E_0 \hat{x}$ $\qquad V = -E_0 x$

$$= -E_0 \rho \cos\phi$$

$$V_o(\rho, \phi) = -E_0 \rho \cos\phi + \sum A_n \rho^{-n} \cos n\phi + B_n \rho^{-n} \sin n\phi$$

Inside conductor V_i = constant. Choose it to be zero because we can.

V_o must be continuous at conductor's surface.

$$V_o(a, \phi) = 0$$

To achieve this $B_n = 0$, $A_n = 0$ except $n=1$

$$V_0 = -E_0 \rho \cos\phi + A_1 \rho^{-1} \cos\phi$$

$$V(a,\phi) = 0 = \left[-E_0 a + \frac{A_1}{a} \right] \cos\phi$$

$$A_1 = +a^2 E_0$$

$$V(\rho,\phi) = -E_0 \rho \cos\phi + \frac{a^2 E_0}{\rho} \cos\phi$$