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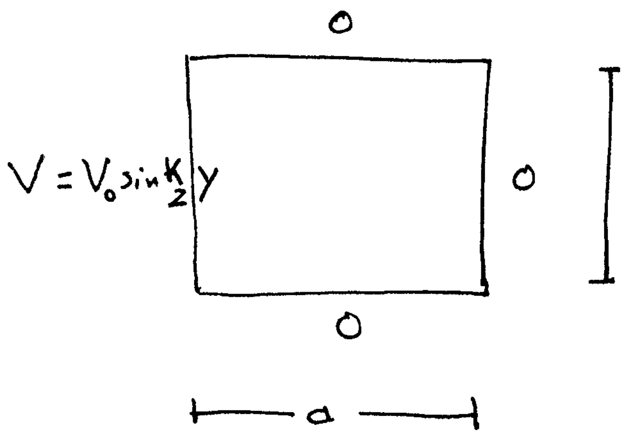


<u>Sln</u> The two dimensional Laplacian separates and produces solutions that look like

$$V(x,y) = \left[e^{kx}, e^{-kx} \right] \times \left[\sin ky, \cos ky \right]$$

<u>Boundary Conditions</u>

1. $V(a,y) = 0$

2. $V(x,0) = 0$

3. $V(x,b) = 0$

4. $V(0,y) = V_0 \sin \frac{2\pi}{b} y$

Apply Boundary Conditions

2. $\Rightarrow$ Eliminate $\cos kx$ solutions

3. $V(x,b)=0 \quad \Rightarrow \quad \sin kb = 0$

$$\Rightarrow \quad k = k_n = \frac{n\pi}{b}$$

General Solution with these restrictions

$$V(x,y) = \sum_n \left(A_n e^{k_n x} + B_n e^{-k_n x}\right) \sin k_n y$$

Apply BC (1)

$$V(a,x) = 0 = \sum_n \left(A_n e^{k_n a} + B_n e^{-k_n a}\right) \sin k_n y$$

By orthogonality, these must be zero term by term

$$A_n e^{k_n a} + B_n e^{-k_n a} = 0$$

$$A_n = - B_n e^{-2k_n a}$$

$$V(x,y) = \sum_n \left(B_n e^{-k_n x} - B_n e^{k_n x} e^{-2k_n a} \right) \sin k_n y$$

Apply BC (4)

$$V(0,y) = V_0 \sin \frac{2\pi}{b} y$$

$$= V_0 \sin k_2 y = \sum_n \left(B_n - B_n e^{-2k_n a} \right) \sin k_n y$$

<u>Apply Orthogonality</u>

$$B_n = 0 \qquad n \neq 2$$

$$B_2 - B_2 e^{-2k_2 a} = V_0$$

$$B_2 = \frac{V_0}{1 - e^{-2k_2 a}}$$

Full Potential

$$V(x,y) = \frac{V_0}{1 - e^{-\frac{4\pi a}{b}}} \left[e^{-k_z x} - e^{k_z x} e^{-2k_z a} \right] \sin \frac{2\pi}{b} y$$

The field is

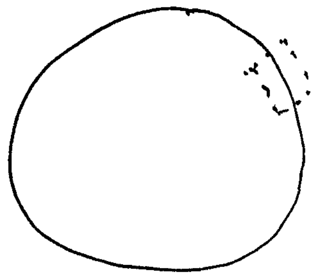
$$\vec{E} = -\nabla V =$$

$$\frac{-V_0}{1 - e^{-2k_z a}} \sin k_z y \left(-k_z e^{-k_z x} - k_z e^{k_z x} e^{-2k_z a} \right) \hat{x}$$

$$- \frac{V_0 k_z}{1 - e^{-2k_z a}} \cos k_z y \left(e^{-k_z x} - e^{k_z x} e^{-2k_z a} \right) \hat{y}$$

The field at the center is just this evaluated at $x = a/2$, $y = b/2$.

(8)



Using a Gaussian pillbox at the surface, ~~the~~

Gauss' law give us

$$\vec{E}_o \cdot \hat{\rho} A - \vec{E}_i \cdot \hat{\rho} A = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

$$\vec{E}_o \cdot \hat{\rho} = -\frac{\partial V_o}{\partial r}\bigg|_a \qquad \vec{E}_i \cdot \hat{\rho} = -\frac{\partial V_i}{\partial \rho}\bigg|_a$$

$$\frac{\partial V_i}{\partial \rho}\bigg|_a - \frac{\partial V_o}{\partial \rho}\bigg|_a = \frac{\sigma}{\varepsilon_0}$$

$$\frac{\partial V_i}{\partial \rho}\bigg|_a = \cancel{4Aa\ 4aa}$$

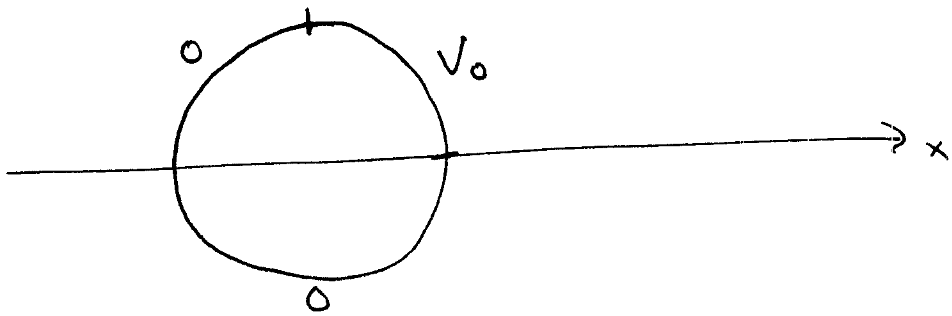
$$= 4Aa^3 \cos 4\phi$$

$$\left.\frac{\partial V_o}{\partial \rho}\right|_a = -\frac{4B}{a^5}\cos 4\phi$$

$$\sigma = \varepsilon_o \left(\left.\frac{\partial V_i}{\partial \rho}\right|_a - \left.\frac{\partial V_o}{\partial \rho}\right|_a \right)$$

$$= \varepsilon_o \cos 4\phi \left(4 A a^3 + \frac{4B}{a^5} \right)$$

(9)



The potential outside is given by

$$V(\rho, \phi) = \sum_n A_n \rho^{-n} \cos n\phi + B_n \sin n\phi \, \rho^{-n}$$

At the surface,

$$V(a, \phi) = \sum_n A_n a^{-n} \cos n\phi + B_n \sin n\phi \, a^{-n}$$

Use orthogonality

$$A_n = \frac{a^n}{\pi} \int_0^{2\pi} d\phi \, V \cos n\phi$$

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} d\phi \, V$$

$$B_n = \frac{a^n}{\pi} \int_0^{2\pi} d\phi \, V \sin n\phi$$

$$A_0 = \frac{1}{2\pi} V_0 \cdot \pi/2$$

$$= \frac{V_0}{4}$$

$$A_n = \frac{a^n V_0}{\pi} \int_0^{\pi/2} \cos n\phi \, d\phi$$

$$= \frac{a^n V_0}{n\pi} \sin n\phi \Big|_0^{\pi/2}$$

$$= \frac{a^n V_0}{n\pi} \sin \frac{n\pi}{2}$$

$$\sin \frac{n\pi}{2} = \begin{array}{ll} 1 & n = 1, 5, 9 \\ -1 & n = 3, 7, 11 \end{array}$$

$$B_n = \frac{a^n V_0}{\pi} \int_0^{\cancel{2}\pi/2} \sin n\phi \, d\phi$$

$$= -\frac{a^n V_0}{n\pi} \cos n\phi \Big|_0^{\pi/2} = \frac{a^n V_0}{n\pi} \left(1 - \cos \frac{n\pi}{2}\right)$$

$$V(\rho, \phi) = \frac{V_0}{4} + \sum_{n=1} \frac{a^n V_0}{n\pi} \sin\frac{n\pi}{2}\, \rho^{-n} \cos n\phi$$

$$+ \frac{a^n V_0}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right) \rho^{-n} \sin n\phi$$

(10) For a sphere,

$$V(r,\theta) = \sum_n A_n r^{-(n+1)} P_n(\cos\theta)$$

At the surface,

$$\int_{-1}^{1} P_m(\cos\theta)\, V(a,\theta)\, d\cos\theta$$

$$= A_m a^{-(m+1)} \cancel{P_m(\cos\theta)} \cdot \frac{2}{2m+1}$$

using $\displaystyle\int_{-1}^{1} P_n(\cos\theta)\, P_m(\cos\theta)\, d\cos\theta = \frac{2}{2n+1}\,\delta_{nm}$

So

$$A_m = \frac{a^{m+1}(2m+1)}{2} \int_{-1}^{1} P_m(\cos\theta)\, V(a,\theta)\, d\cos\theta$$

The lower limit is $\cos\theta = -1 \Rightarrow \theta = \pi$

The upper limit is $\cos\theta = 1 \Rightarrow \theta = 0$

To pick up the part of the potential that is non-zero we want to integrate to ~~$\cos\frac{\pi}{4} = \pi/4$~~ $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$A_n = \frac{a^{n+1}(2n+1)}{2} V_0 \int_{\frac{1}{\sqrt{2}}}^{1} P_n(\cos\theta)\, d\cos\theta$$

$$= \frac{a^{n+1}(2n+1)}{2} V_0 \int_{\frac{1}{\sqrt{2}}}^{1} P_n(x)\, dx$$

Find first two non-zero terms $P_0 = 1$

$P_1 = x$

$$A_0 = \frac{V_0}{2} \int_{\frac{1}{\sqrt{2}}}^{1} dx = \frac{V_0}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$A_1 = \frac{3}{2} a V_0 \int_{\frac{1}{r_2}}^{1} x \, dx$$

$$= \frac{3}{4} a V_0 \; x^2 \Big|_{\frac{1}{r_2}}^{1}$$

$$A_1 = \frac{3}{8} V_0 a$$