PHYS 3414 - Electricity and Magnetism- Homework Set 2

Chapter 2 - Electric Fields

Due 12:30pm Wednesday January 30, 2008 at the beginning of class.

Good's Problems

All problems must be worked from first principles by starting from Coulomb's or Gauss' Law.

2.8

20pts 2.12

10pts 2.14

2.16

20pts 2.18

2.22

2.28

2.30

20pts 2.34

Additional Problems

A1 part(a) 10pts

Problem A1 Re-work problem A1 of the previous homework using the electric potential. If you already did it using the potential, congrats.

20pts **Problem A2** The electric potential of a point dipole was given in lecture as $V = kp\cos(\theta)/r^2$. Compute the electric field. Sketch the field.

Problem A3 Evaluate the integral

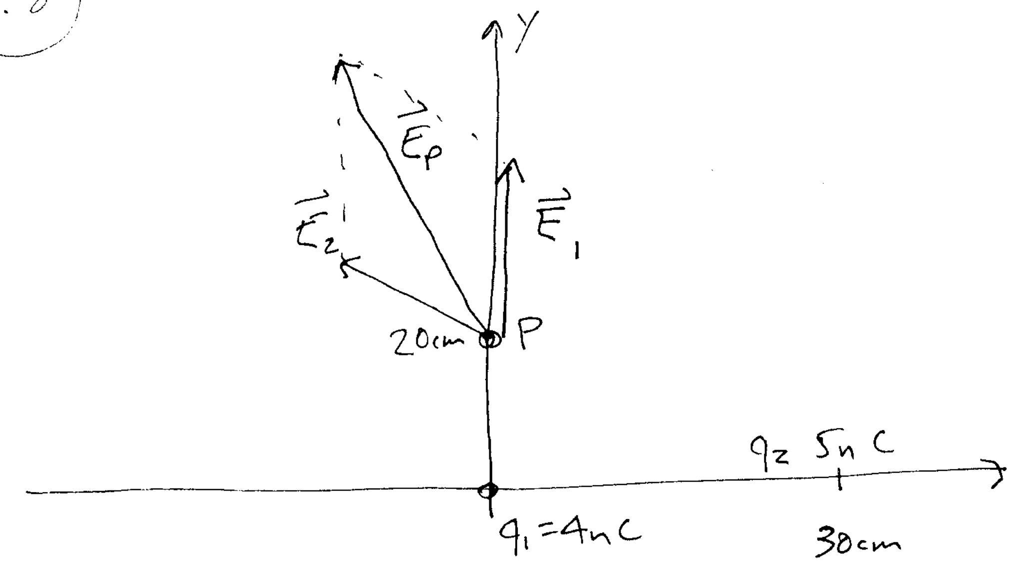
$$\int_V ((x-5)^2 e^{-3(z-5)} \nabla \cdot \left(\frac{\hat{r}}{r^2}\right) dv$$

over a sphere of radius R centered at the origin.

UPII Problems

No UPII problems this time because many of the above are UPII problems.

2.8



Field of q_1

$$\vec{E}_{1P} = \frac{k q_1}{r_{1P}^2} \hat{r}_{1P} = \frac{\left(9\times10^9 \frac{Nm^2}{C^2}\right)\left(4\times10^{-9}C\right)}{(0.2m)^2} (0,1,0)$$

$$= 900 N/C \ \hat{y}$$

Field of q_2

$$\vec{r}_{1P} = (-30cm, 20cm, 0)$$

$$r_{1P} = \sqrt{1300} \ cm = 10\sqrt{13} \ cm$$

$$\hat{r}_{2P} = \frac{\vec{r}_{2P}}{r_{2P}} = \left(-\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}, 0\right)$$

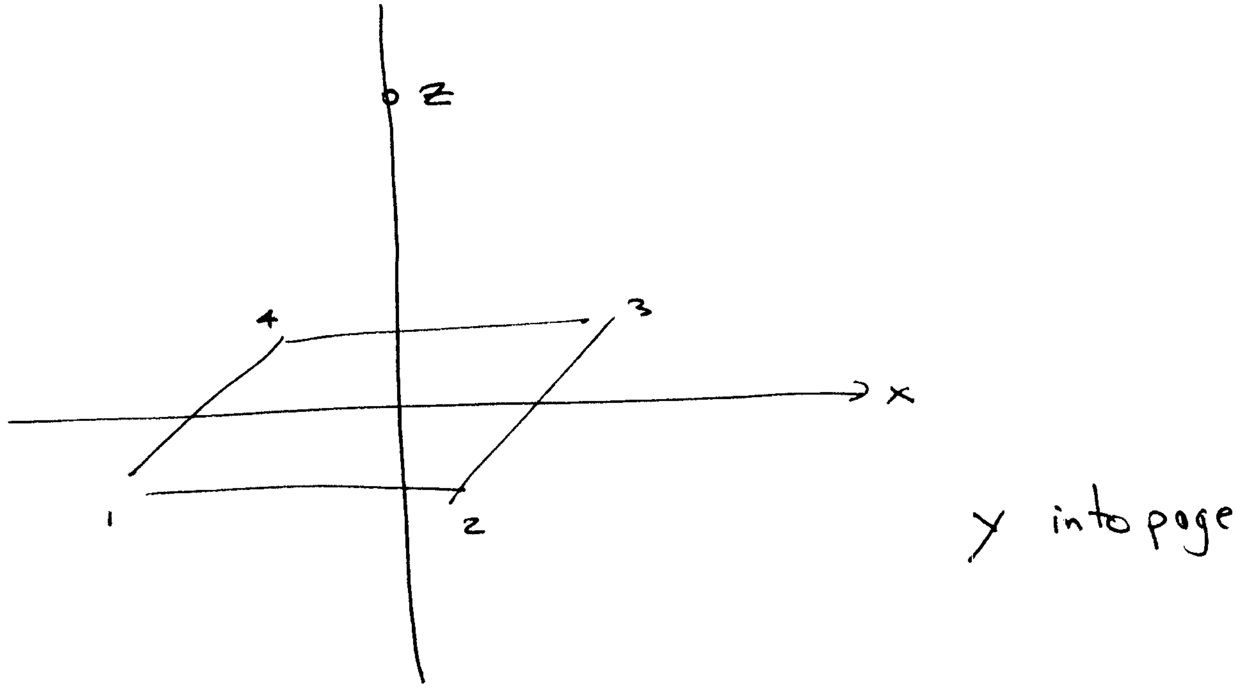
$$\vec{E}_{2P} = \frac{k q_2}{r_{2P}^2} \hat{r}_{2P} = \frac{\left(9\times10^9 \frac{Nm^2}{C^2}\right)\left(5\times10^{-9}C\right)}{\left(\sqrt{1300}\times10^{-2}m\right)^2} \hat{r}_{2P}$$

$$\vec{E}_{2P} = 346\,N/C\left(\frac{-3}{\sqrt{13}}, \frac{2}{\sqrt{13}}, 0\right)$$

$$= \left(-300\,N/C\,,\; 200\,N/C\,,\; 0\right)$$

$$\vec{E}_P = \vec{E}_{1P} + \vec{E}_{2P} = \left(-300\,N/C\,,\; 1100\,N/C\,,\; 0\right)$$

(2.12)



For one of the sides, say 1, the component of $\vec{E}$ in the $\hat{z}$ direction is

$$E_1 = \int_{-s/2}^{s/2} dx' \, E(x', -s/2, z) \cos\theta'$$

$$\cos\theta' = \vec{r}'' \cdot \hat{z}$$

$$\vec{r}'' = \cancel{x} \quad \vec{r} - \vec{r}'$$

$$= (0, 0, z) - \left(x', -\frac{s}{2}, 0\right)$$

$$\vec{r}'' = \left(-x',\ s/2,\ z\right)$$

$$r'' = \sqrt{x'^2 + s^2/4 + z^2}$$

$$dq = \lambda\, dx' \qquad \cos\theta' = \frac{z}{\sqrt{x'^2 + s^2/4 + z^2}}$$

$$\vec{E} = \int_{-s/2}^{s/2} \frac{k\lambda\, dx'\, z}{\left(x'^2 + s^2/4 + z^2\right)^{3/2}}\,\hat{z}$$

$$= \frac{8sk\lambda z}{\sqrt{2s^2+4z^2}\,\left(s^2+4z^2\right)}\,\hat{z}$$

The total field is 4 times this

$$\vec{E} = \frac{32sk\lambda z}{\sqrt{2s^2+4z^2}\,\left(s^2+4z^2\right)}\,\hat{z}$$

$$\lim_{z\to 0} \quad \vec{E} \to 0 \quad \checkmark$$

$$\lim_{z\to \infty} \quad \vec{E} \Rightarrow \frac{32sk\lambda z}{(2z)(4z^2)}\,\hat{z} \quad \text{since } z \gg s$$

$$= \frac{k(4s\lambda)}{z^2}\,\hat{z} \quad \checkmark \quad \text{since } Q = 4s\lambda$$

> $int\left(\dfrac{k \cdot lambda \cdot z}{\left(xp^2 + \left(\frac{s}{2}\right)^2 + z^2\right)^{\left(\frac{3}{2}\right)}},\ xp\right);$

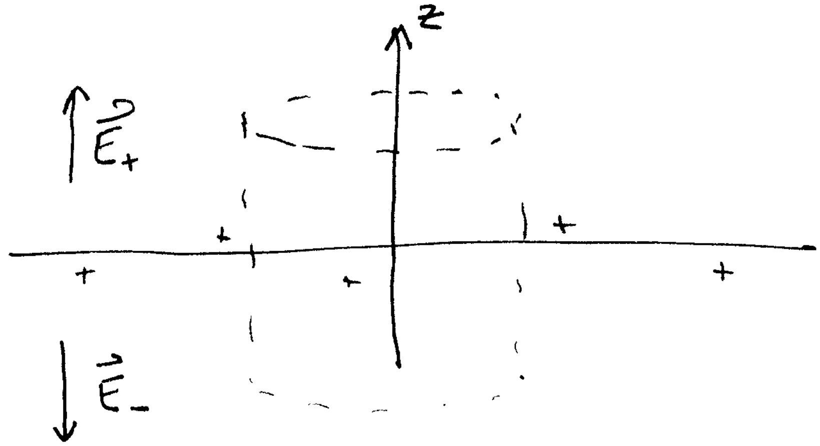
$$\frac{8\,xp\,k\,\lambda\,z}{\sqrt{4\,xp^2 + s^2 + 4\,z^2}\,\left(s^2 + 4\,z^2\right)} \qquad (1)$$

> $int\left(\dfrac{k \cdot \lambda \cdot z}{\left(xp^2 + \left(\frac{s}{2}\right)^2 + z^2\right)^{\left(\frac{3}{2}\right)}},\ xp = -\left(\frac{s}{2}\right)..\left(\frac{s}{2}\right)\right)$ *assuming* $z > 0,\ s > 0;$

$$\frac{8\,s\,k\,\lambda\,z}{\sqrt{2\,s^2 + 4\,z^2}\,\left(s^2 + 4\,z^2\right)} \qquad (2)$$

>

2.14 Let the charge be in the x-y plane



Use a Gaussian cylinder of end area A. The flux out of the surface for $\vec{E}_+$ and $\vec{E}_-$ above is

$$\Phi = E_+A + E_-A = \frac{Q_{enc}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

Note, the flux out of the curved sides is zero because $\vec{E} \perp \hat{n}$.

By reflection symmetry, $E_+ = E_-$.

$$\Phi = 2E_+A = \frac{\sigma A}{\epsilon_0}$$

$$\vec{E}_+ = \frac{\sigma}{2\epsilon_0}\hat{z} = \frac{4\times10^{-9}\,C/m^2}{2\cdot\left(8.85\times10^{-12}\,\frac{C^2}{Nm^2}\right)}\hat{z}$$

$$= 200\,\frac{N}{C}\hat{z} \quad (226\,N/C \text{ but 1 sig fig})$$

$$\vec{E}_- = -200\,N/C\,\hat{z}$$

2.16 With the Gaussian surface drawn in figure 2.30,

If $\rho < R$, $\quad Q_{enc} = \cancel{\pi\rho^2}\ \pi r^2 l \rho$

$$\Phi = EA = 2\pi r l E = \frac{Q_{enc}}{\varepsilon_0}$$

$$\vec{E} = \frac{Q_{enc}}{2\pi r l \varepsilon_0}\hat{\rho} = \frac{\pi r^2 l \rho}{2\pi r l \varepsilon_0}\hat{\rho}$$

$$= \frac{r\rho}{2\varepsilon_0}\hat{\rho}$$

charge density

If $r > R$,

$$Q_{enc} = \pi R^2 l \rho$$

$$\vec{E} = \frac{Q_{enc}}{2\pi r l \varepsilon_0}\hat{\rho} = \frac{\pi R^2 l \rho}{2\pi r l \varepsilon_0}\hat{\rho}$$

$$= \frac{R^2 \rho}{2\varepsilon_0 r}\hat{\rho}$$

The total charge per unit length is $\lambda = \dfrac{Q}{l} = \dfrac{\pi R^2 \rho l}{l}$

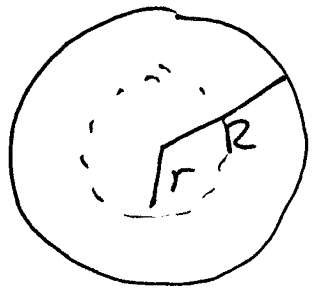
$$= \pi R^2 \rho$$

In terms of the total charge per unit length

If $r < R$ $\quad \vec{E} = \frac{r\rho}{2\varepsilon_0}\hat{\rho} = \frac{\lambda r}{2\pi R^2 \varepsilon_0}\hat{\rho}$

If $r > R$ $\quad \vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{\rho}$ as expected.

2.18



Using a spherical Gaussian surface, the flux out of the surface is

$$\Phi = EA = 4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_0} \text{ by Gauss'}$$

$$\vec{E} = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}\hat{r}$$

If $r < R$, $\quad Q_{enc} = \int_0^r 4\pi r^2 \rho(r)\, dr$

using the shell method or by performing the angular integrals by brute force.

$$Q_{enc} = \int_0^r 4\pi r^2 \left(\frac{\sigma}{r}\right) dr$$

$$= 4\pi\sigma \int_0^r r\, dr$$

$$= 2\pi\sigma r^2$$

$$\vec{E} = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{2\pi\sigma r^2}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{\sigma}{2\varepsilon_0}\hat{r}$$

If $r > R$, the charge enclosed is

$$Q_{enc} = \int_0^R 4\pi r^2 \rho(r)\, dr = 2\pi\sigma R^2 \equiv Q$$

$$\vec{E} = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{2\pi\sigma R^2}{4\pi\varepsilon_0 r^2}\hat{r}$$

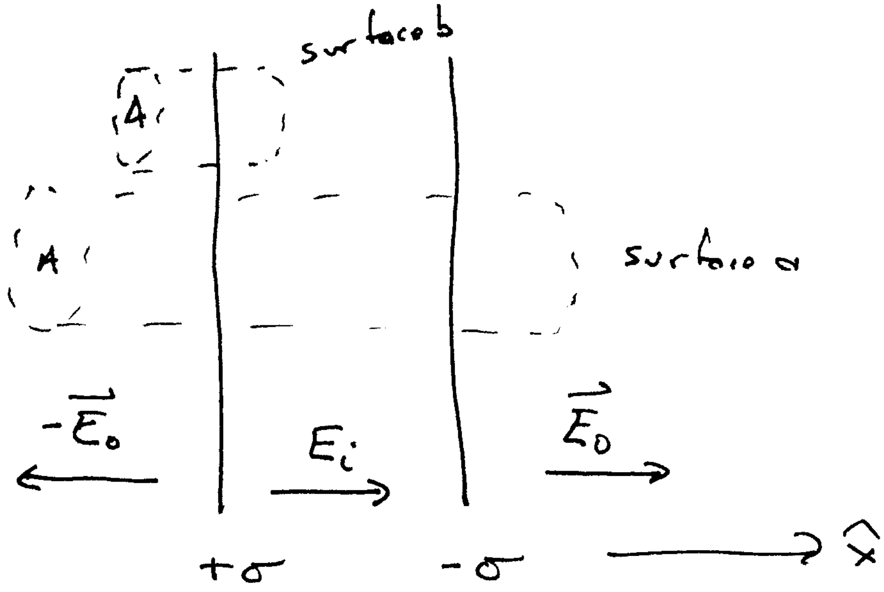
$$= \frac{\sigma R^2}{2\varepsilon_0 r^2}\hat{r}$$

Or in terms of the total charge Q

$r < R$ $\quad \vec{E} = \dfrac{Q}{4\pi\varepsilon_0 R^2}\hat{r}$

$r > R$ $\quad \vec{E} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$ as expected.

2.22



The total flux out of surface a is zero, since the charge enclosed is zero. Since the field of the planes does not decay with distance, the same E_o results if the planes are pushed together to form a single plane. This plane has reflection symmetry so the outer fields are equal and opposite, therefore

$$\vec{E}_o = 0.$$

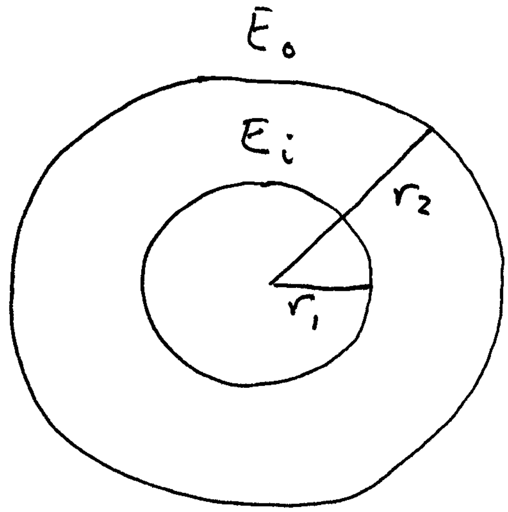
Applying Gauss' Law to cylindrical surface b,

$$\Phi = E_i A = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

$$\vec{E}_i = \frac{\sigma}{\varepsilon_0}\hat{x}$$

For the field given, $\sigma = \varepsilon_0 E_i = \left(8.8\times10^{-12}\frac{C^2}{Nm^2}\right)\left(10^6 N/C\right) = 8.85\times10^{-6}\frac{C}{m^2}$

2-24



Between the spheres, only the charge on the inner sphere contributes and the field is

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

Outside both spheres, the total charge enclosed by a spherical Gaussian surface is zero, so $\vec{E_o} = 0$

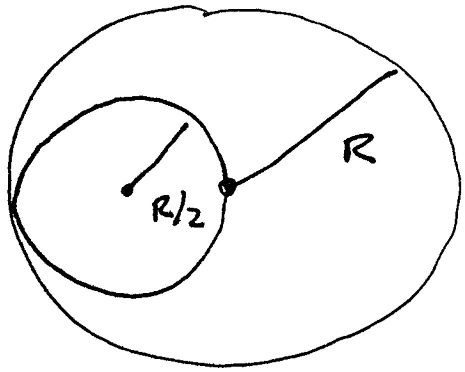
At $r = r_1 = 1\,\text{cm}$,

$$\vec{E} = \frac{KQ}{r^2}\hat{r} = \frac{\left(9\times10^9 \frac{Nm^2}{C^2}\right)\left(2\times10^{-9}C\right)}{(0.01\,m)^2}\hat{r}$$

$$= 18\times10^4\,N/C\ \hat{r}$$

$$= 1.8\times10^5\,N/C\ \hat{r}$$

2.28



Let γ = volume charge density given as ρ in the problem.

Let the z-axis be the axis through the center of the cavity. The field in the cavity is the sum of the field of a cylinder of radius $R/2$ with charge density $-\gamma$ and axis down the z-axis and a solid cylinder with radius R and charge density γ with axis parallel to the z-axis through the point $R/2\,\hat{x}$.

The field of the $-\gamma$ cylinder is

$$\Phi = 2\pi\rho l E = \frac{Q_{enc}}{\epsilon_0} \qquad Q_{enc} = -\gamma\pi\rho^2 l$$

$$\vec{E}_{cav} = -\frac{\gamma\rho}{2\epsilon_0}\hat{\rho}$$

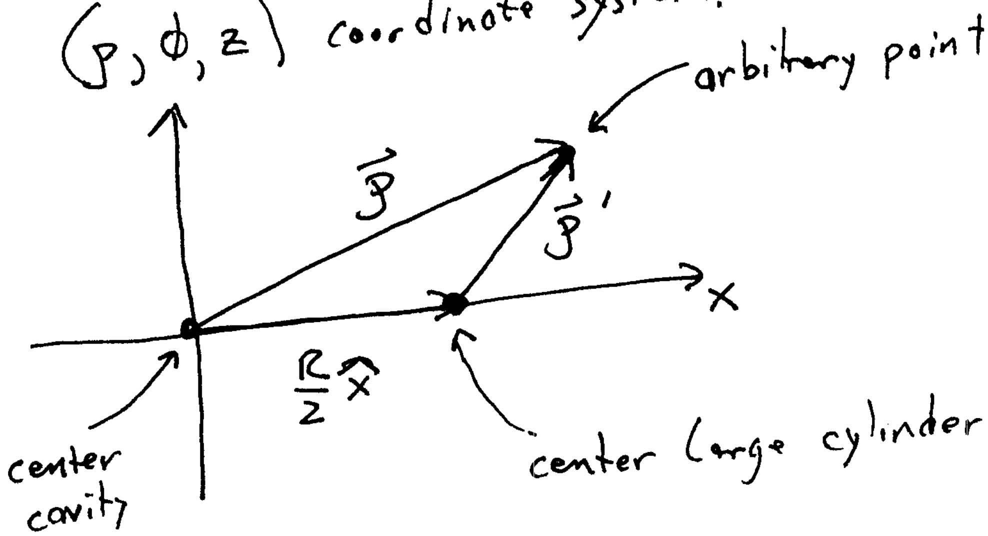
The field of the $+\gamma$ cylinder about its axis is

$$\vec{E}_{cyl} = \frac{\gamma \rho'}{2\varepsilon_0} \hat{\rho}'$$

where the $'$ indicates coordinates with respect to the center of the large cylinder.

$$\vec{E} = \frac{-\gamma \rho}{2\varepsilon_0} \hat{\rho} + \frac{\gamma \rho'}{2\varepsilon_0} \hat{\rho}' = \frac{-\gamma}{2\varepsilon_0} \vec{\rho} + \frac{\gamma}{2\varepsilon_0} \vec{\rho}'$$

We now have to express ρ' and $\hat{\rho}'$ in the (ρ, ϕ, z) coordinate system.

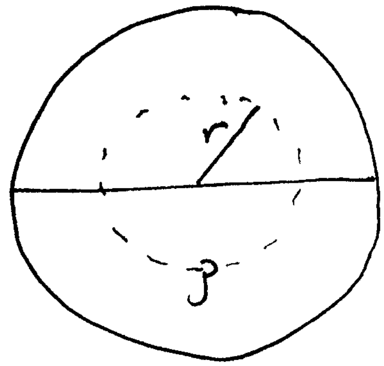


$$\vec{\rho} = \vec{\rho}' + \frac{R}{2}\hat{x} \qquad \vec{\rho}' = \vec{\rho} - \frac{R}{2}\hat{x}$$

$$\vec{E} = \frac{-\gamma \vec{\rho}}{2\varepsilon_0} + \frac{\gamma}{2\varepsilon_0}\left(\vec{\rho} - \frac{R}{2}\hat{x}\right)$$

$$= -\frac{\gamma R}{4\varepsilon_0}\hat{x}$$

2.34



Compute field.

The flux out of a Gaussian surface of radius r is $\Phi = 4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_0}$

The charge inside the surface is $\frac{4}{3}\pi\rho r^3$

$$4\pi r^2 E = \frac{\frac{4}{3}\pi\rho r^3}{\varepsilon_0}$$

$$\vec{E} = \frac{\rho r}{3\varepsilon_0}\hat{r}$$

The force on a small element of charge is $d\vec{F} = dq\,\vec{E}$

The total force is $\vec{F} = \int d\vec{F} = \int_{\text{hemisphere}} \vec{E}\,dq$

$$dq = \rho r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\vec{F} = \int_0^R dr \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \; \frac{\rho^2}{3\varepsilon_0} \left(r^3 \sin\theta \right) \hat{r}$$

$$\hat{r} = \left(\sin\theta\cos\phi ,\; \sin\theta\sin\phi ,\; \cos\theta \right)$$

$$F_x = \frac{\rho^2}{3\varepsilon_0} \int_0^R dr \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi \; r^3 \sin^2\theta \cos\phi$$

$= 0$ since we integrate $\cos\phi$ from $0 \to 2\pi$

Likewise, $F_y = 0$.

$$F_z = \frac{\rho^2}{3\varepsilon_0} \int_0^R dr \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi \; r^3 \sin\theta\cos\theta$$

$$= \frac{2\pi\rho^2}{3\varepsilon_0} \int_0^R r^3 dr \int_0^{\pi/2} \sin\theta\cos\theta \, d\theta$$

$u = \sin\theta$

$du = \cos\theta \, d\theta$

$\int u \, du$

$= \frac{u^2}{2}$

$$= \frac{2\pi\rho^2}{3\varepsilon_0} \int_0^R r^3 dr \left[-\frac{1}{2}\cos^2\theta \right]_0^{\pi/2}$$

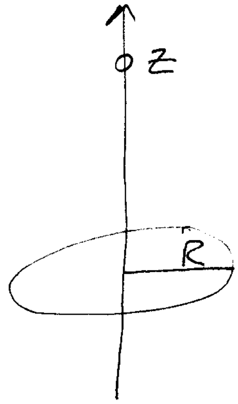
$\not{=}$

$$F_z = +\frac{\pi \rho^2}{3\varepsilon_0}\int_0^R r^3\,dr$$

$$= \frac{\pi \rho^2}{3\varepsilon_0}\,\frac{R^4}{4} = \frac{\pi \rho^2 R^4}{3\varepsilon_0}$$

$$\vec{F} = \frac{\pi \rho^2 R^4}{12\varepsilon_0}\,\hat{z}$$

A1



(a) All the charge is a distance $\sqrt{R^2+z^2}$ from $(0,0,z)$

$$V(z) = \frac{Q}{4\pi\varepsilon_0 d} = \frac{2\pi R\lambda}{4\pi\varepsilon_0\sqrt{z^2+R^2}}$$

$$= \frac{R\lambda}{2\varepsilon_0\sqrt{z^2+R^2}}$$

$$\vec{E}(z) = -\frac{dV}{dz}\hat{z} = \frac{R\lambda z}{2\varepsilon_0(z^2+R^2)^{3/2}}\hat{z}$$

(b) <u>Potential of Disk</u>

$$V(z) = \int_0^R \frac{r\sigma\, dr}{2\varepsilon_0\sqrt{z^2+r^2}} \qquad \lambda = \sigma dr$$

$$\cancel{= \left[\frac{1}{2}\ln(z^2) + \ln\left(R+\sqrt{R^2+z^2}\right)\right]_0 \frac{}{2\varepsilon_0}}$$

Maple (A1)

> $int\left(\dfrac{r}{\left(r^2+z^2\right)^{\left(\frac{1}{2}\right)}},\ r=0..R\right)$ *assuming* $z>0$;

$$-z+\sqrt{R^2+z^2} \tag{1}$$

> $-\,diff(\%,\ z)$;

$$1-\frac{z}{\sqrt{R^2+z^2}} \tag{2}$$

> $int\left(\dfrac{r}{\left(r^2+z^2\right)^{\left(\frac{1}{2}\right)}},\ r=0..R\right)$ *assuming* $z<0$;

$$z+\sqrt{R^2+z^2} \tag{3}$$

> $-\,diff(\%,\ z)$;

$$-1-\frac{z}{\sqrt{R^2+z^2}} \tag{4}$$

This ends the solution of parts (a) and (b). Below is where part (d) went south.

> $int\left(\left(R^2+(z-zp)^2\right)^{\left(\frac{1}{2}\right)},\ zp=-L..L\right)$;

$$\frac{1}{2}\sqrt{L^2+2\,z\,L+R^2+z^2}\;z+\frac{1}{2}\sqrt{L^2+2\,z\,L+R^2+z^2}\;L-\frac{1}{2}\,R^2\ln\Big(-z-L+\sqrt{L^2+2\,z\,L+R^2+z^2}\Big)-\frac{1}{2}\sqrt{L^2-2\,z\,L+R^2+z^2}\;z+\frac{1}{2}\sqrt{L^2-2\,z\,L+R^2+z^2}\;L+\frac{1}{2}\,R^2\ln\Big(-z+L+\sqrt{L^2-2\,z\,L+R^2+z^2}\Big) \tag{5}$$

> $simplify(\%\ )$;

$$\frac{1}{2}\sqrt{L^2+2\,z\,L+R^2+z^2}\;z+\frac{1}{2}\sqrt{L^2+2\,z\,L+R^2+z^2}\;L-\frac{1}{2}\,R^2\ln\Big(-z-L+\sqrt{L^2+2\,z\,L+R^2+z^2}\Big)-\frac{1}{2}\sqrt{L^2-2\,z\,L+R^2+z^2}\;z+\frac{1}{2}\sqrt{L^2-2\,z\,L+R^2+z^2}\;L+\frac{1}{2}\,R^2\ln\Big(-z+L+\sqrt{L^2-2\,z\,L+R^2+z^2}\Big) \tag{6}$$

> $-\,diff(\%,\ z)$;

$$-\frac{1}{4}\,\frac{z\,(2\,L+2\,z)}{\sqrt{L^2+2\,z\,L+R^2+z^2}}-\frac{1}{2}\sqrt{L^2+2\,z\,L+R^2+z^2}-\frac{1}{4}\,\frac{L\,(2\,L+2\,z)}{\sqrt{L^2+2\,z\,L+R^2+z^2}}+\frac{1}{2}\,\frac{R^2\left(-1+\frac{1}{2}\,\frac{2\,L+2\,z}{\sqrt{L^2+2\,z\,L+R^2+z^2}}\right)}{-z-L+\sqrt{L^2+2\,z\,L+R^2+z^2}}+\frac{1}{4}\,\frac{z\,(-2\,L+2\,z)}{\sqrt{L^2-2\,z\,L+R^2+z^2}}+\frac{1}{2}\sqrt{L^2-2\,z\,L+R^2+z^2}-\frac{1}{4}\,\frac{L\,(-2\,L+2\,z)}{\sqrt{L^2-2\,z\,L+R^2+z^2}} \tag{7}$$

$$-\frac{1}{2}\frac{R^2\left(-1+\frac{1}{2}\frac{-2L+2z}{\sqrt{L^2-2zL+R^2+z^2}}\right)}{-z+L+\sqrt{L^2-2zL+R^2+z^2}}$$

> *simplify*(%);

$$-\frac{1}{\sqrt{L^2-2zL+R^2+z^2}\,\sqrt{L^2+2zL+R^2+z^2}}\Big(2\sqrt{L^2-2zL+R^2+z^2}\,zL + \sqrt{L^2-2zL+R^2+z^2}\,z^2+\sqrt{L^2-2zL+R^2+z^2}\,L^2+R^2\sqrt{L^2-2zL+R^2+z^2} + 2\sqrt{L^2+2zL+R^2+z^2}\,zL-\sqrt{L^2+2zL+R^2+z^2}\,z^2-\sqrt{L^2+2zL+R^2+z^2}\,L^2 - R^2\sqrt{L^2+2zL+R^2+z^2}\Big) \tag{8}$$

> *int*$\left(\left(R^2+(z-zp)^2\right)^{\left(\frac{1}{2}\right)}, zp\right)$;

$$\frac{1}{4}(-2z+2zp)\sqrt{R^2+z^2-2zzp+zp^2}+\frac{1}{2}R^2\ln\left(-z+zp+\sqrt{R^2+z^2-2zzp+zp^2}\right) \tag{9}$$

> $-$ *diff*(%, z);

$$\frac{1}{2}\sqrt{R^2+z^2-2zzp+zp^2}-\frac{1}{8}\frac{(-2z+2zp)(2z-2zp)}{\sqrt{R^2+z^2-2zzp+zp^2}} -\frac{1}{2}\frac{R^2\left(-1+\frac{1}{2}\frac{2z-2zp}{\sqrt{R^2+z^2-2zzp+zp^2}}\right)}{-z+zp+\sqrt{R^2+z^2-2zzp+zp^2}} \tag{10}$$

> *simplify*(%);

$$\sqrt{R^2+z^2-2zzp+zp^2} \tag{11}$$

>

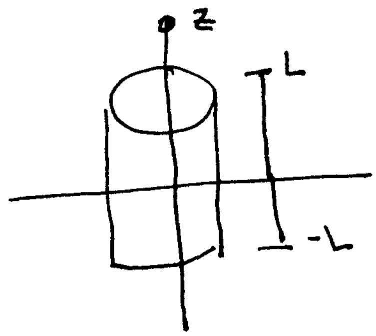
The above is what we got using the field method. Not worth the extra effort.

$$V(z) = \frac{\sigma}{2\varepsilon_0}\left(-|z| + \sqrt{R^2+z^2} \right)$$

$$\vec{E}(z) = -\frac{dV}{dz}\hat{z} = \frac{\sigma}{2\varepsilon_0}\,\mathrm{sgn}(z)\,\hat{z} - \frac{\sigma z\,\hat{z}}{2\varepsilon_0\sqrt{R^2+z^2}}$$

End of required

<u>Volume</u> $(z>L)$



$\sigma = \rho\, dz'$

$$V(z) = \frac{\rho}{2\varepsilon_0}\int_{-L}^{L}\left(-|z-z'| + \sqrt{R^2+(z-z')^2}\right)dz'$$

<u>Do integrals individually.</u>

$$\int_{-L}^{L} -|z-z'|\,dz' = \int_{-L}^{L} z-z'\,dz' \qquad \text{since } z>L$$

$$= \left(zz' - \frac{z'^2}{2}\right)_{-L}^{L} = 2Lz$$

$$\int_{-L}^{L}\sqrt{R^2+(z-z')^2}\,dz' =$$

$$\frac{1}{4}(-2z+2z')\sqrt{R^2+z^2-2zz'+z'^2} + \frac{1}{2}R^2\ln\left(-z+z'+\sqrt{R^2+z^2-2zz'+z'^2}\right)$$

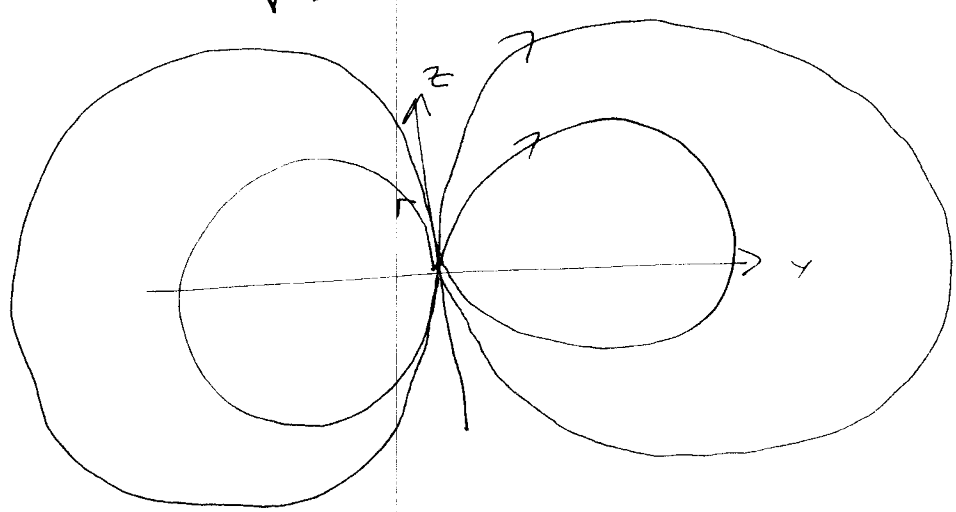
At this point, it become obvious that for this part, the potential method was not an improvement. The simpler field result was hiding. The attached maple sheet extracts it.

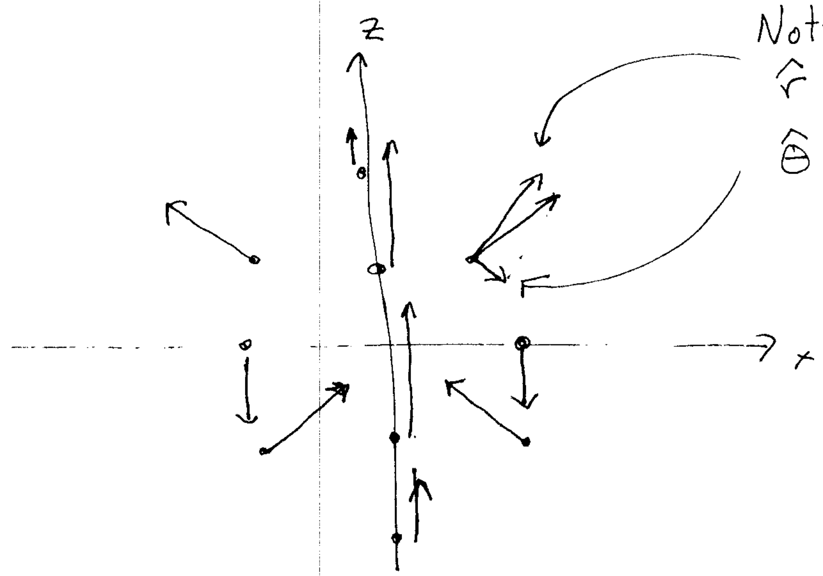
A2

$$V = \frac{k p \cos\theta}{r^2}$$

$$\vec{E} = -\nabla V = -\left(\frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi} \right)$$

$$= \frac{2pk\cos\theta}{r^3}\hat{r} + \frac{pk\sin\theta}{r^3}\hat{\theta}$$





Note
$\hat{r}$ component
$\hat{\theta}$ component

A3

$$\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta(\vec{r})$$

$$4\pi \int_V (x-5)^2 e^{-3(z-5)} \delta(\vec{r})\, dv$$

$$= 4\pi(-5)^2 e^{15} = 25e^{15} \cdot 4\pi$$

$$= 100\pi e^{15}$$