PHYS 3414 - Electricity and Magnetism- Homework Set 3

Chapter 3 - Magnetic Fields

Due 12:30pm Wednesday February 6, 2008 at the beginning of class.

Good's Problems

All problems must be worked from first principles by starting from the Biot-Savart Law or Ampere's Law

3.6

3.8 20

3.10

3.12 20

3.14

3.16 20

3.18 (skip solid angle)

3.20

~~3.22~~ Skip

3.24

3.26 The field is sketched in the problem

~~3.36~~ Skip.

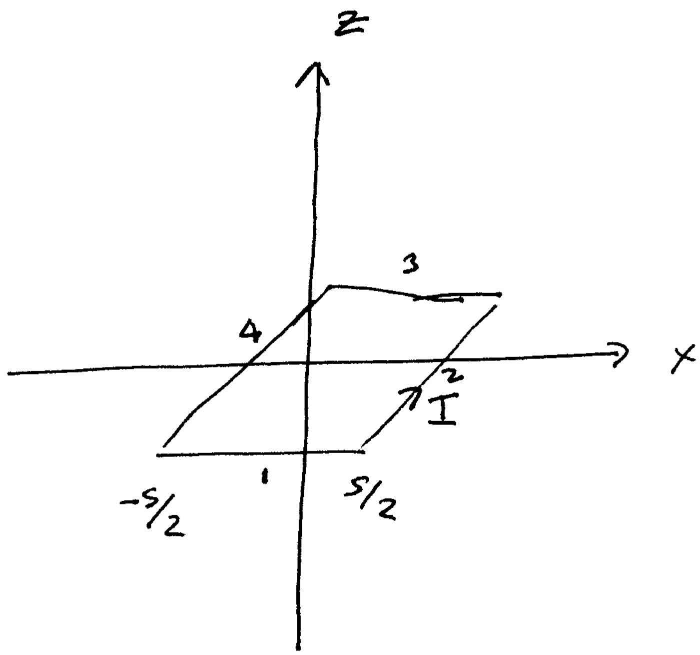
Additional Problems

20 **Problem A1** Let the system in parts (a) and (b) of problem A1 of Homework 1 rotate about its axis with an angular velocity ω. Calculate the magnetic field along the axis.

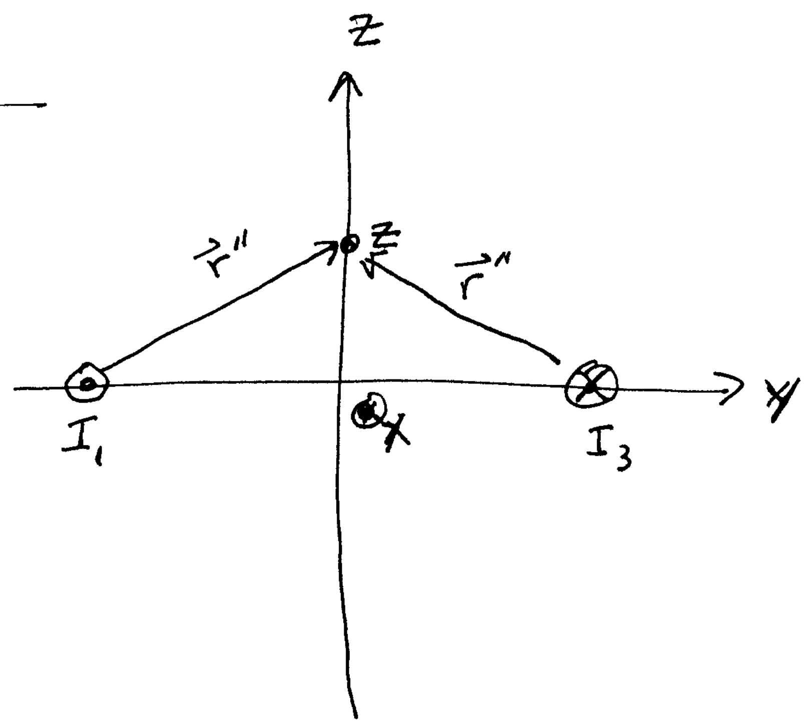
UPII Problems

No UPII problems this time because many of the above are UPII problems.

(3.6)



Side view



By the $\pi/2$ rotation symmetry, only the z-component survives.

Do one side

$$\vec{B}_1 = \frac{\mu_0 I}{4\pi} \int_{\text{side 1}} \frac{d\vec{l}' \times \hat{r}''}{r''^2} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}' \times \vec{r}''}{r''^3}$$

$$d\vec{l}' = +dx'\hat{x}$$

$$\vec{r} = (0, 0, z) \qquad \vec{r}' = (x', -s/2, 0)$$

$$\vec{r}'' = (-x', s/2, z)$$

$$r'' = \sqrt{x'^2 + s^2/4 + z^2}$$

$$d\vec{l}' \times \vec{r}'' = dx' \left(\tfrac{s}{2}\, \hat{x} \times \hat{y} + z\, \hat{x} \times \hat{z} \right)$$

$$= \frac{s}{2} dx'\, \hat{z} - z\, dx'\, \hat{y}$$

$$\vec{B}_1 = -\frac{\mu_0 I}{4\pi} z \left[\int_{-s/2}^{s/2} \frac{dx'}{(x'^2 + (s/2)^2 + z^2)^{3/2}} \right] \hat{y}$$

$$+ \frac{\mu_0 I}{4\pi} \left(\frac{s}{2} \right) \left[\int_{-s/2}^{s/2} \frac{dx'}{(x'^2 + (s/2)^2 + z^2)^{3/2}} \right] \hat{z}$$

We only need the z component.

$$\int_{-s/2}^{s/2} \frac{dx'}{\left(x'^2+\left(\frac{s}{2}\right)^2+z^2\right)^{3/2}} = 2\int_0^{s/2} \frac{dx'}{\left(x'^2+\left(\frac{s}{2}\right)^2+z^2\right)^{3/2}}$$

$$= \frac{16x'}{\sqrt{4x'^2+s^2+4z^2}\,(s^2+4z^2)}\Bigg|_0^{s/2}$$

$$= \frac{8s}{\sqrt{2s^2+4z^2}\,(s^2+4z^2)}$$

$$= \frac{s}{\left(z^2+\left(\frac{s}{2}\right)^2\right)\sqrt{z^2+2\left(\frac{s}{2}\right)^2}}$$

$$B_{1z} = \frac{\mu_0 I}{8\pi}\,\frac{s^2}{\left(z^2+\left(\frac{s}{2}\right)^2\right)\left(\sqrt{z^2+2\left(\frac{s}{2}\right)^2}\right)}$$

Total Field is 4 times this

$$\vec{B} = \frac{\mu_0 I}{2\pi}\,\frac{s^2}{\left(z^2+\left(\frac{s}{2}\right)^2\right)\sqrt{z^2+2\left(\frac{s}{2}\right)^2}}$$

$\vec{B}(0) = 1.1\times 10^{-4}\,T\,\hat{z}$

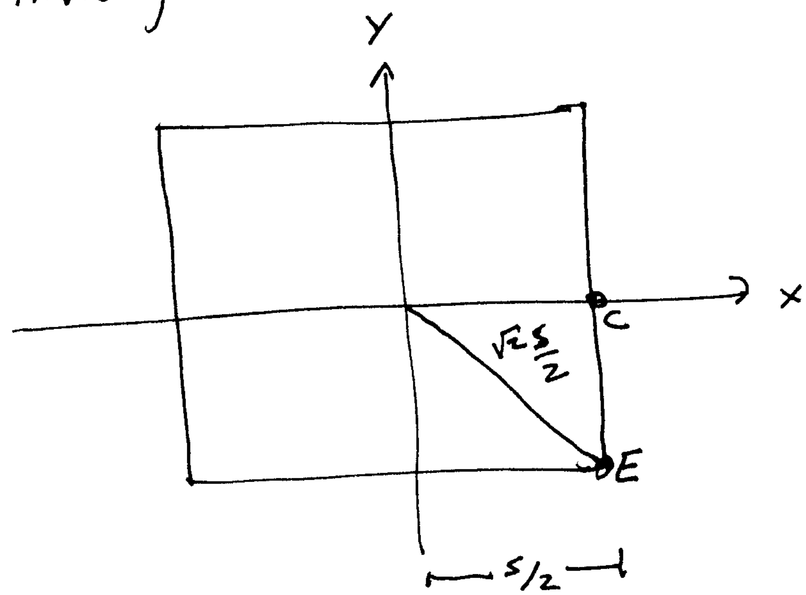
> $int\left(\frac{1}{\left(xp^2+\left(\frac{s}{2}\right)^2+z^2\right)^{\left(\frac{3}{2}\right)}},\ xp\right);$

$$\frac{8\,xp}{\sqrt{4\,xp^2+s^2+4\,z^2}\,\left(s^2+4\,z^2\right)} \qquad (1)$$

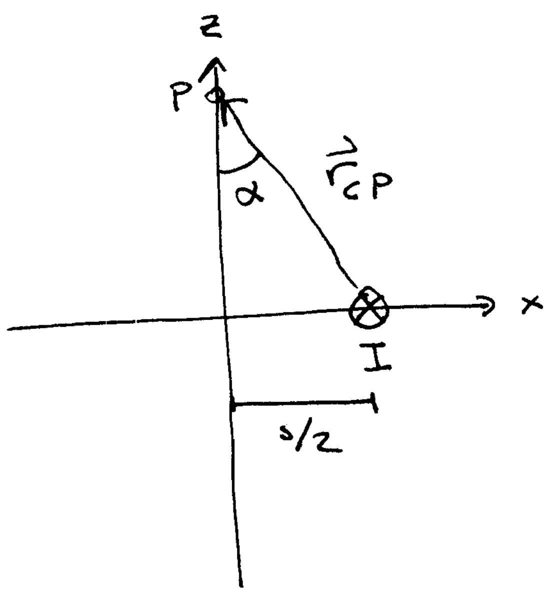
>

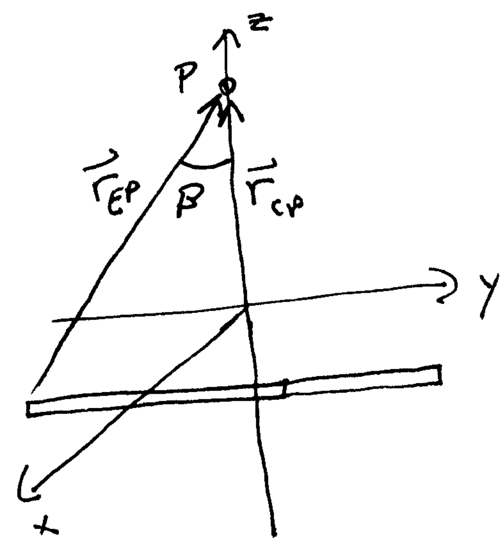
>

We're consistently getting the same mess. We should investigate.



Consider only the side containing the points C and E.





I have introduced two angles that may be interesting.

α - The angle between $\vec{r}_{CP}$ and the z axis.

β - The angle between $\vec{r}_{CP}$ and $\vec{r}_{EP}$.

Calculate α

$$\sin\alpha = \frac{s/2}{r_{CP}} = \frac{s/2}{\sqrt{z^2+(s/2)^2}}$$

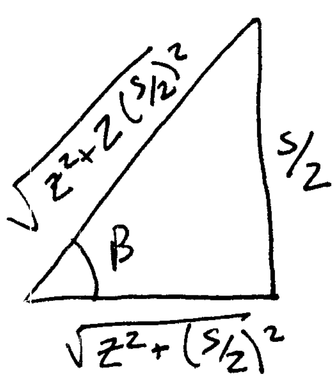
$$\vec{r}_{CP} = \left(0, -s/2, z\right) \qquad r_{CP} = \sqrt{z^2+(s/2)^2}$$

$$\cos B = \frac{\vec{r}_{CP}\cdot\vec{r}_{EP}}{|\vec{r}_{CP}||\vec{r}_{EP}|}$$

$$\vec{r}_{EP} = \left(-s/2, -s/2, z\right) \qquad r_{EP} = \sqrt{2(s^2/2)+z^2}$$

$$\cos B = \frac{z^2+(s/2)^2}{\sqrt{z^2+(s/2)^2}\,\sqrt{z^2+2(s/2)^2}} = \frac{\sqrt{z^2+(s/2)^2}}{\sqrt{z^2+2(s/2)^2}}$$

Find $\sin B$



$$\sin B = \frac{s/2}{\sqrt{z^2+2(s/2)^2}}$$

Re-write our magnetic field

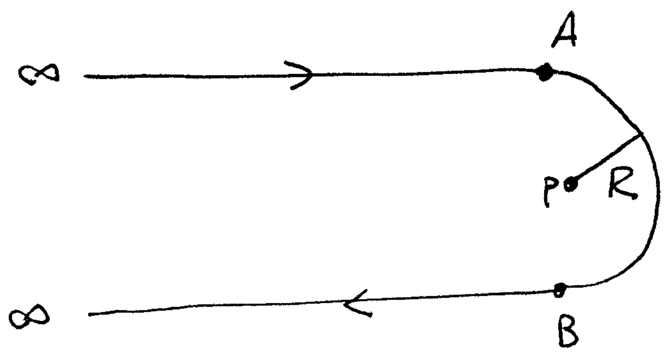
$$\vec{B} = \frac{\mu_0 I}{2\pi} \frac{s^2}{\left(z^2 + \left(s/2\right)^2\right)\sqrt{z^2 + 2\left(s/2\right)^2}} \hat{z}$$

$$= \frac{2\mu_0 I}{\pi r_{cp}} \frac{s/2}{\sqrt{z^2 + \left(s/2\right)^2}} \frac{s/2}{\sqrt{z^2 + 2\left(s/2\right)^2}} \hat{z}$$

$$= \frac{2\mu_0 I}{\pi r_{cp}} \sin\alpha \sin\beta \hat{z}$$

So the mess we keep getting involves various angles in the geometry of the problem.

3.8



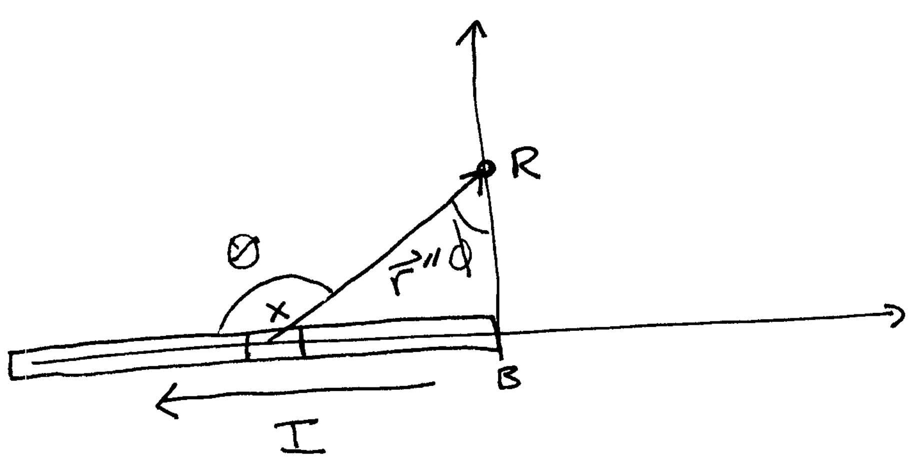
Unfortunately, by the RHR all segments contribute.

$$\vec{B}_P = \vec{B}_{\infty A} + \vec{B}_{AB} + \vec{B}_{B\infty}$$

$$= 2\vec{B}_{\infty A} + \vec{B}_{AB}$$

All fields are into the page by the RHR.

Calculate $\vec{B}_{\infty B}$



$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \hat{r}''}{r''^2} \qquad |d\vec{B}| = \frac{\mu_0 I}{4\pi} \frac{dl'}{r''^2} \sin\theta$$

Work out the angles

$$\boxed{\begin{aligned}\frac{x}{R} &= \tan\phi \\ dx &= R\sec^2\phi\, d\phi\end{aligned}}$$

$$\boxed{\begin{aligned}\pi - \theta + \phi &= \pi/2 \\ \theta &= \pi/2 + \phi \\ \sin\theta &= \cos\phi\end{aligned}}$$

$$\frac{R}{r''} = \cos\phi$$

$$|d\vec{B}| = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{0}\frac{dx\sin\theta}{r''^2} = \frac{\mu_0 I}{4\pi}\int_{\pi/2}^{0}\frac{(R\sec^2\phi\, d\phi)(\cos\phi)}{(R/\cos\phi)^2}$$

$$= \frac{\mu_0 I}{4\pi R}\int_{\pi/2}^{0} d\phi\cos\phi = \frac{\mu_0 I}{4\pi R}\sin\phi\Big|_{\pi/2}^{0}$$

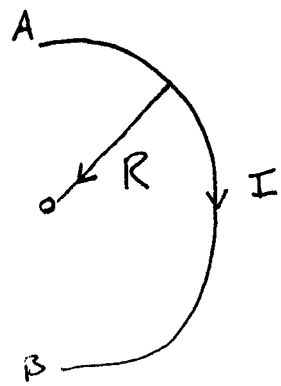
$$= -\frac{\mu_0 I}{4\pi R}$$

(Note, gives infinite wire if upper limit $\pi/2$)

Discard the negative sign because we already worked out the direction.

$$\vec{B}_{\infty} = \frac{\mu_0 I}{4\pi R} \text{ into page}$$

Calculate field of circle



$$|d\vec{B}| = \frac{\mu_0 I}{4\pi} \frac{d\ell \sin\theta}{R^2} \qquad \theta = 90^\circ$$

$$|\vec{B}| = \frac{\mu_0 I}{4\pi R^2} \int d\ell$$

$$= \frac{\mu_0 I}{4\pi R^2} \pi R = \frac{\mu_0 I}{4R}$$

Total Field

$$\vec{B} = 2\vec{B}_{\infty} + \vec{B}_{AB}$$

$$= \left(\frac{\mu_0 I}{2\pi R} + \frac{\mu_0 I}{4R} \right) = \frac{\mu_0 I}{R}\left(\frac{1}{2\pi} + \frac{1}{4} \right) \text{ into page}$$

$$= \frac{(4\pi \times 10^{-7}\,\mathrm{Tm/A})(1\,\mathrm{A})}{0.01\,\mathrm{m}} \left(\frac{1}{2\pi} + \frac{1}{4} \right) \text{ into page}$$

$= 5\times10^{-5}$ into page $\approx$ Earth's field

3.10

Magnetic force

$$F_m = qvB = ma_c = \frac{mv^2}{r}$$

$$\frac{qBr}{m} = v$$

$$v = \frac{(1.6\times10^{-19}C)(1.5T)(0.07m)}{1.67\times10^{-27}kg}$$

$$= 1\times10^{7}\,m/s$$

3.12 Cyclotron frequency is the frequency of the orbit of a charged particle in a magnetic field.

$$\omega = \frac{2\pi}{T} \qquad \text{where } T \text{ is the period}$$

$$v = \frac{2\pi r}{T}$$

$$F_m = qvB = ma_c = \frac{mv^2}{r}$$

$$v = \frac{qBr}{m} = \frac{2\pi r}{T} = \omega r$$

$$\boxed{\omega = \frac{qB}{m}}$$

Proton $$\omega = \frac{(1.6\times10^{-19}\,C)(1.2\,T)}{1.67\times10^{-27}\,kg} = 1.1\times10^{8}\,Hz$$

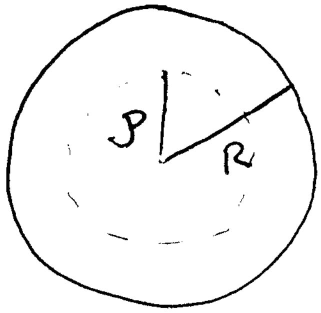
Electron $$\omega = \frac{(1.6\times10^{-19}\,C)(1.2\,T)}{9.11\times10^{-31}\,kg} = 2\times10^{11}\,Hz$$

3.14 Infinite wire

$$B = \frac{\mu_0 I}{2\pi R} = \frac{(4\pi \times 10^{-7} \, Tm/A)(4A)}{2\pi (0.03m)}$$

$$= 2.7 \times 10^{-5} \, T$$

3.16



Total Current

$$I = \int_S J\,da = \int_0^R (2\pi\rho\, d\rho)\, K/\rho = \int_0^{2\pi} d\phi \int_0^R d\rho\, \rho \left(\frac{K}{\rho}\right)$$

$$= 2\pi K R$$

B inside $(\rho < R)$ Use circular Amperian path.

$$I_{enc} = \int_0^r 2\pi\rho\, d\rho\, \frac{K}{\rho} = 2\pi K \rho$$

Ampere's Law

$$\oint \vec{B}\cdot d\vec{l} = \int B\, dl = 2\pi\rho\, B(\rho) = \mu_0 I_{enc}$$

$$|\vec{B}| = \frac{\mu_0 I_{enc}}{2\pi\rho} = \frac{\mu_0 (2\pi K \rho)}{2\pi\rho} = \mu_0 K$$

$$= \frac{\mu_0 I}{2\pi R}$$

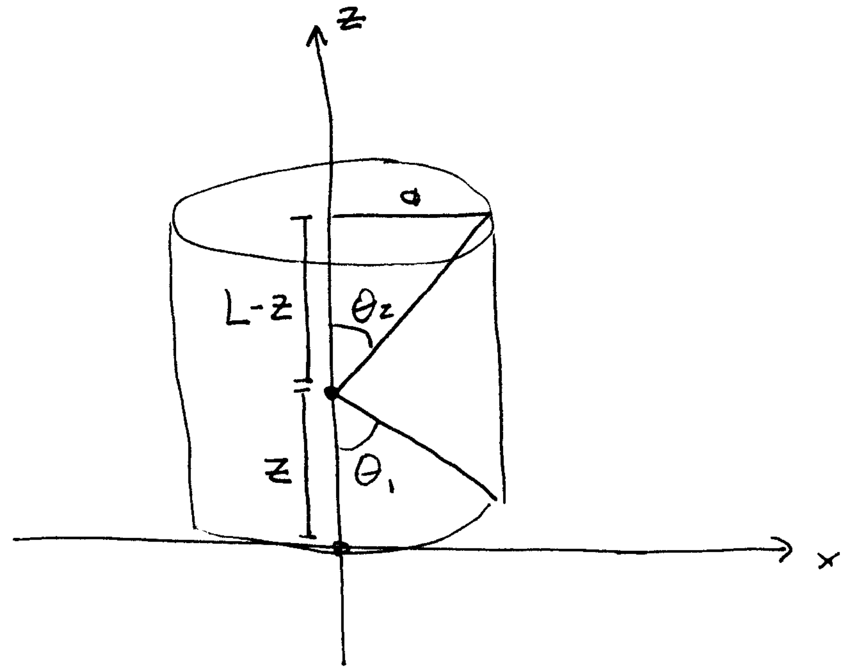
B outside $(\rho > R)$

$$I_{enc} = I$$

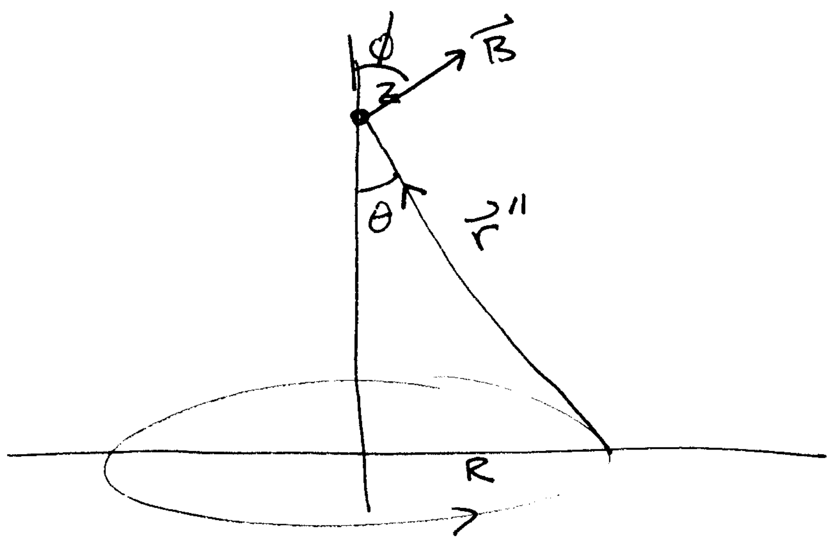
$$|\vec{B}| = \frac{\mu_0 I}{2\pi\rho} = \frac{\mu_0 2\pi K R}{2\pi\rho}$$

$$= \frac{\mu_0 K R}{\rho}$$

(3.18)



The magnetic field of a circular wire in the x-y plane is



For any small element $d\vec{l}$,

$$|d\vec{B}| = \frac{\mu_0 I}{4\pi} \frac{|d\vec{l} \times \hat{r}''|}{r''^2}$$

$$|d\vec{l} \times \hat{r}''| = dl \sin\theta = dl$$

$$r''^2 = (R^2+z^2)$$

$$= \frac{\mu_0 I \, dl}{4\pi(R^2+z^2)}$$

Only the z-component contributes, so we need to multiply by $\cos\phi = \sin\theta = \dfrac{R}{\sqrt{R^2+z^2}}$

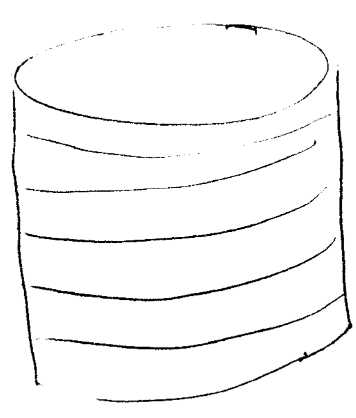
$$|d\vec{B}| = \frac{\mu_0 I R \, dl}{4\pi (R^2+z^2)^{3/2}}$$

$$\vec{B} = \frac{\mu_0 I R}{4\pi (R^2+z^2)^{3/2}} \int_{\circ} dl$$

$$= \frac{\mu_0 I R^2}{2 (R^2+z^2)^{3/2}} \hat{z}$$

Check $z=0$ $\quad \dfrac{\mu_0 I}{2R}\hat{z}$ ✓

Now, integrate the wire over the cylinder. ~~The~~ Slice the cylinder into circles of thickness dz'. The effective current $I = dz' K$



$$\vec{B} = \int_0^L \frac{dz' \, K \mu_0 R^2}{2\left(R^2+(z-z')^2\right)^{3/2}} \hat{z}$$

$$= \frac{K \mu_0 R^2 \hat{z}}{2} \int_0^L \frac{dz'}{\left(R^2+(z-z')^2\right)^{3/2}}$$

<u>Change variables</u> $\quad u = z - z' \quad du = -dz'$

$$\vec{B} = -\frac{K \mu_0 R^2 \hat{z}}{2} \int_z^{z-L} \frac{du}{(R^2+u^2)^{3/2}}$$

$$= \frac{K \mu_0 R^2 \hat{z}}{2} \int_{-z}^{L-z} \frac{du}{(R^2+u^2)^{3/2}}$$

Let $u = R\tan\theta \qquad R^2(1+\tan^2\theta) = R^2\sec^2\theta$

$du = R\sec^2\theta \, d\theta$

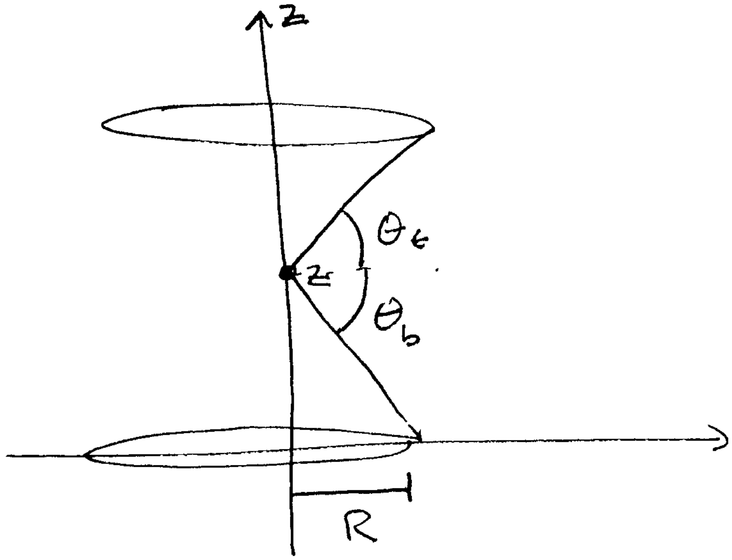
$$\vec{B} = \frac{K \mu_0 R^2 \hat{z}}{2} \int_{-z}^{L-z} \frac{R\sec^2\theta \, d\theta}{(R^2\sec^2\theta)^{3/2}}$$

$$= \frac{K \mu_0 R^2}{2R^2} \hat{z} \int_{-z}^{L-z} \cos\theta \, d\theta$$

Fix the limits

$$\frac{L-z}{R} = \tan\theta_t$$

$$\frac{-z}{R} = -\tan\theta_b$$



$$\vec{B} = \left(\frac{K\mu_0}{2}\hat{z}\right)\left(\sin\theta_t + \sin\theta_b\right) = \frac{K\mu_0}{2}\hat{z}\int_{-\theta_b}^{\theta_t}\cos\theta\, d\theta$$

$\lim L \to \infty$ $\theta_t, \theta_b \to \pi/2$, $B \to K\mu_0$ = infinite solenoid ✓

Solid angle $\Omega = \int_{\theta_2}^{\pi-\theta_1} \sin\theta\, d\theta\, d\phi = 2\pi\left(\cos(\pi-\theta_1) - \cos\theta_2\right)$

$$\cancel{= 2\pi\left(\cos\theta_t - \cos\theta_b\right.}$$

$$= 2\pi\left(\cos\theta_1 + \cos\theta_2\right)$$

$$= 2\pi\left(\sin\theta_t + \sin\theta_b\right)$$

3.20

From previous problem,

$$B = \frac{\mu_0 I_{inner}}{2\pi r} = \frac{(4\pi \times 10^{-7}\,\mathrm{Tm/A})(10\,\mathrm{A})}{2\pi(0.01\,\mathrm{m})}$$

$$= 2\times 10^{-4}\,\mathrm{T}$$

3.24 $\vec{B} = x\hat{y}$

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x & 0 \end{vmatrix}$$

$$= \hat{z}(1-0)$$

$$= \hat{z}$$

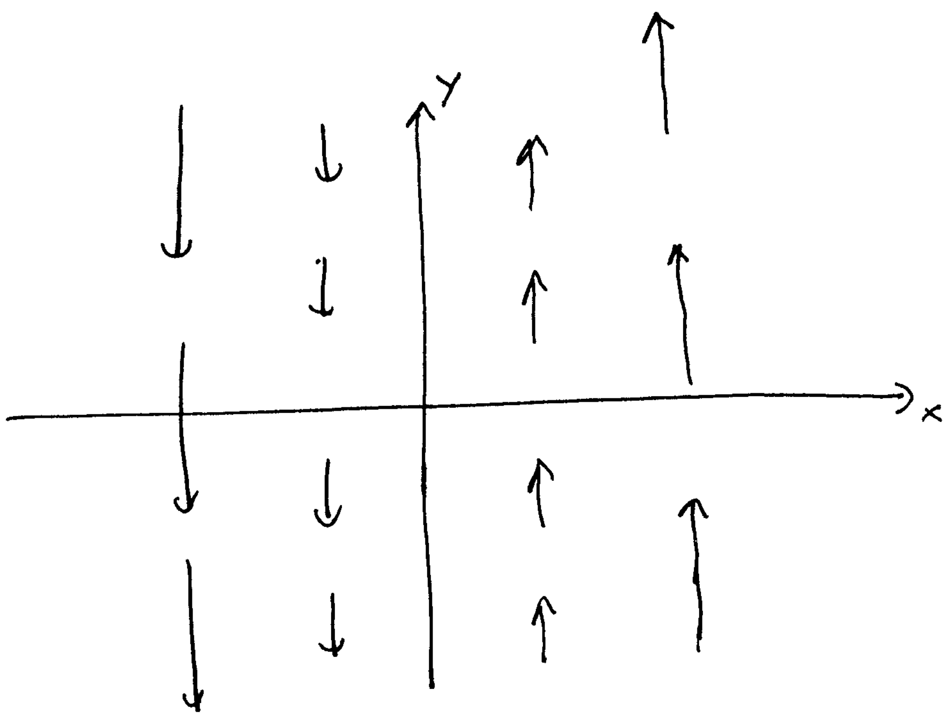
Current Density

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

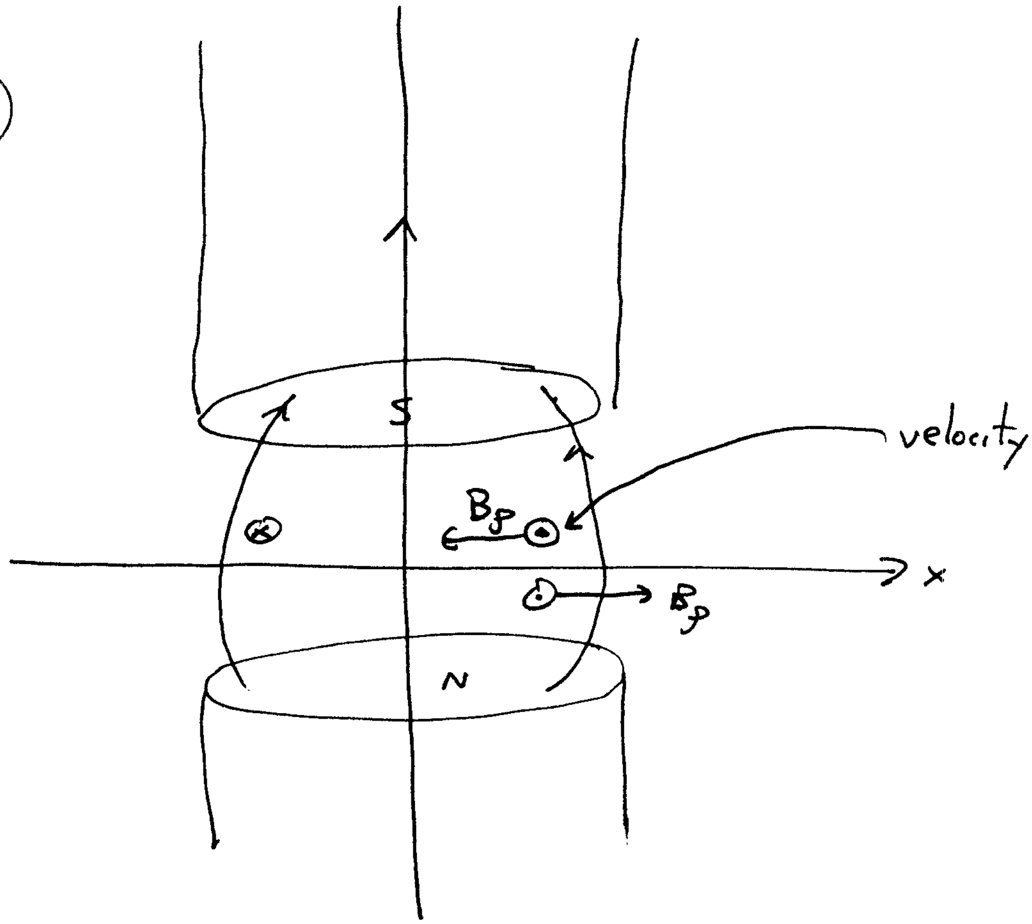
$$\vec{J} = \frac{1}{\mu_0}\hat{z}$$

Field is Possible if $\nabla \cdot \vec{B} = 0$, which it is.

Sketch Field



3.26



We know $B_\rho(0)=0$ by symmetry.

For a + particle to move in a circular orbit, it must experience an inward force. Using $\vec{F}=q\,\vec{v}\times\vec{B}$ this requires the velocity drawn $\vec{v}=-v_0\hat{\phi}$

For a particle with this velocity to feel a force toward the x-y plane, the particle must be in a magnetic field s.t. $B_\rho<0$ when $z>0$ and $B_\rho>0$ when $z<0$.

Since $B_\rho(0)=0$, this implies $\frac{\partial B_\rho}{\partial z}<0$ produces the required effect.

Since there are no currents or changing electric fields present, Ampere's law becomes

$$\nabla \times \vec{B} = 0$$

The ϕ component is

$$\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho} = 0$$

$$\frac{\partial B_\rho}{\partial z} = \frac{\partial B_z}{\partial \rho} < 0 \quad \text{(Given)}$$

So the field has the required property.

(A1) (a) The spinning ring produces a circular current

$$I = \lambda v = \lambda \omega R$$

The field of a ring of current was worked out in 3.18.

$$\vec{B} = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}\hat{z} = \frac{\mu_0 \lambda \omega R^3}{2(R^2+z^2)^{3/2}}\hat{z}$$

(b) The current density of the disk is

$$K = \sigma v = \sigma \omega r$$

The effective current of a thin strip dr is

$$dI = \sigma \omega r\, dr$$

The field of the disk is the

$$\vec{B} = \hat{z}\int_0^R \frac{\mu_0 r^2\, dI}{2(r^2+z^2)^{3/2}} = \hat{z}\int_0^R \frac{\mu_0 \sigma \omega r^3\, dr}{2(r^2+z^2)^{3/2}}$$

$$=$$

> $int\left(\frac{r^3}{\left(r^2+z^2\right)^{\left(\frac{3}{2}\right)}},\ r\right);$

$$\frac{2\,z^2+r^2}{\sqrt{r^2+z^2}} \tag{1}$$

>

$$\vec{B} = \frac{\mu_0 \sigma \omega \hat{z}}{2} \left(\frac{2z^2 + r^2}{\sqrt{r^2+z^2}} \right) \Bigg|_0^R$$

$$= \frac{\mu_0 \sigma \omega \hat{z}}{2} \left(\frac{2\cancel{R}^2 + R^2}{\sqrt{R^2+z^2}} - \frac{2z^2}{\sqrt{z^2}} \right)$$

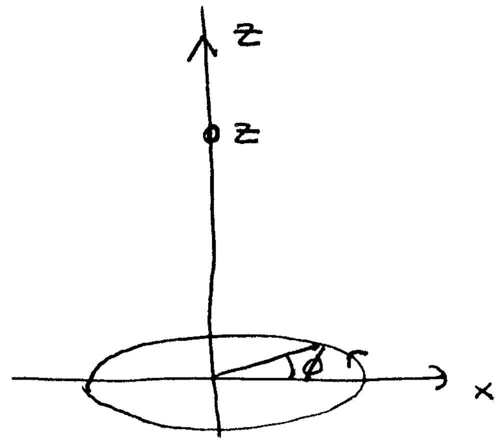
$$= \frac{\mu_0 \sigma \omega \hat{z}}{2} \left(\frac{2z^2 + R^2}{\sqrt{R^2+z^2}} - 2z \right)$$

<u>Units</u> $\mu_0 \frac{C}{m^2} \frac{1}{s} m = \frac{\mu_0 A}{m}$

<u>Solenoid</u> $[B] = T = \left[\mu_0 I \frac{N}{\ell}\right]$

$$= \frac{\mu_0 A}{m} \quad \checkmark$$

Re-work the ring in cylindrical coordinates



$\vec{r} = (0, 0, z)$ $\qquad$ $\vec{r}\,' = (R, 0, 0)$

$\vec{r}\,'' = (-R, 0, z)$ $\qquad$ $d\vec{l} = R\,d\phi\,\hat{\phi}$

$= (0, R\,d\phi, 0)$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_\phi \frac{d\vec{l} \times \vec{r}\,''}{r''^3}$$

$$d\vec{l} \times \vec{r}\,'' = R\,d\phi\,\hat{\phi} \times (-R\hat{\rho} + z\hat{z})$$

We only want the piece in the z-direction

$\hat{\rho} \times \hat{\phi} = \hat{z} \Rightarrow \hat{\phi} \times \hat{\rho} = -\hat{z}$

$d\vec{l} \times \vec{r}\,'' = R^2\,d\phi\,\hat{z}$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \hat{z} \int_0^{2\pi} \frac{R^2}{(z^2+R^2)^{3/2}} d\phi$$

$$= \frac{\mu_0 I}{2} \frac{R^2}{(z^2+R^2)^{3/2}} \hat{z}$$

The same field we calculated by taking the cosine explicitly.