PHYS 3414 - Electricity and Magnetism- Homework Set 6

Chapter 6 and 7 - Potential and Multipole Expansions

Due 5:30pm Monday March 10, 2008.

Good's Problems

6.2

6.6 20pts

6.8 20pts

6.10

6.20 20pts

6.22 20pts

6.26

6.28

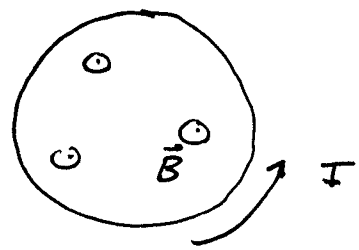
6.36

7.4

7.6

7.10 20pts

6.2



$d\vec{F} = I\,d\vec{l} \times \vec{B}$ gives an outward force.

The pressure is
$$P = \frac{1}{2\mu_0} B^2$$
$$= \frac{1}{2\mu_0} (K\mu_0)^2$$
$$= \frac{\mu_0 K^2}{2}$$
$$= \frac{\mu_0 (N'I)^2}{2} =$$
$$= \frac{(4\pi \times 10^{-7}\,Tm/A)(10^4 \frac{1}{m} \cdot 2A)^2}{2}$$
$$= 8\pi \times 10^1\ N/m^2$$
$$= 80\pi\ N/m^2$$

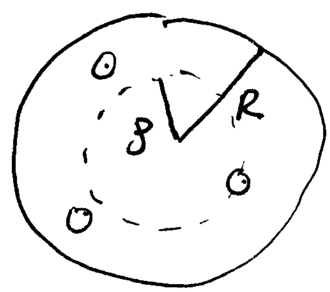
Note, a simple energy argument would give you an inward force, but you have to account for the energy to maintain the current.

6.6

The field of an infinite solenoid is $B = \mu_0 N' I$ for $\rho < R$ and $B = 0$ for $\rho > R$ where R is the radius of the solenoid

The integral of the vector potential around a closed path is the magnetic flux

$$\Phi_m = \oint \vec{A} \cdot d\vec{l}$$



The vector potential will be circular around the axis. $(\nabla \cdot \vec{A} = 0)$
The magnetic flux through a curve of radius $\rho < a$ is

$$\Phi = \pi \rho^2 B = \oint \vec{A} \cdot d\vec{l} = 2\pi\rho A$$

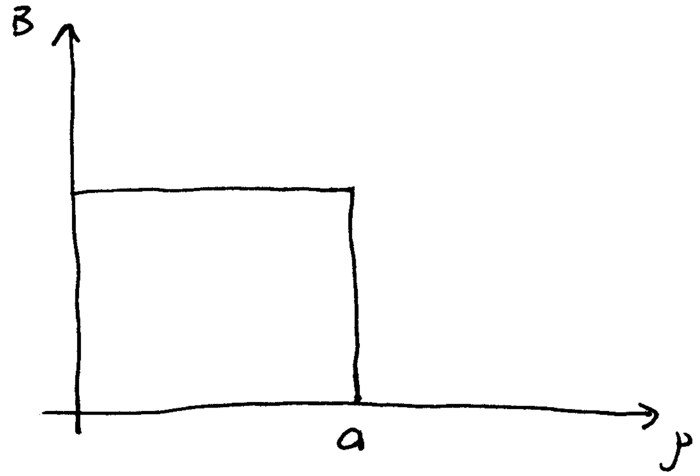
$$\vec{A}(\rho) = \frac{\rho B}{2} \hat{\phi} = \frac{\rho \mu_0 N' I}{2} \hat{\phi}$$

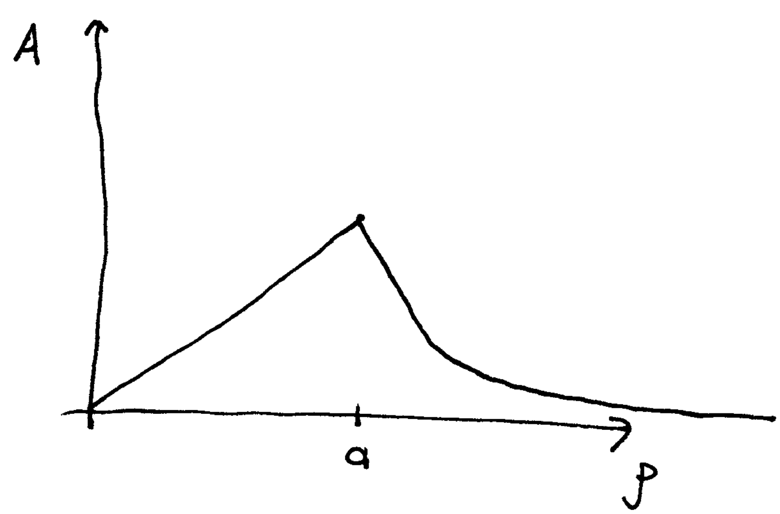
The magnetic field outside solenoid is zero. The vector potential is

$$\Phi = \pi a^2 B = \oint \vec{A} \cdot d\vec{l} = 2\pi\rho A(\rho)$$

$$\vec{A}(\rho) = \frac{a^2 B}{2\rho}\hat{\phi} = \frac{a^2 N' I \mu_0}{2\rho}\hat{\phi}$$

~~Inside~~





6.8

Spherical Shell. Field outside shell (Gauss)

$$\vec{E}_o = \frac{Q}{4\pi\varepsilon_o r^2}\hat{r}$$

The energy density is

$$u = \frac{1}{2}\varepsilon_o E_o^2$$

$$= \frac{1}{2}k^2\varepsilon_o \frac{Q^2}{r^4}$$

The total energy is

$$U = \int dv\, u = \int_R^\infty dr \int_0^{2\pi} r\sin\theta\, d\phi \int_0^\pi r\, d\theta\, u$$

$$= \int_R^\infty 4\pi r^2 u\, dr = \int_R^\infty 4\pi r^2 \left(\frac{1}{2}k^2\varepsilon_o \frac{Q^2}{r^4}\right) dr$$

$$= \frac{1}{2}kQ^2 \int_R^\infty \frac{dr}{r^2}$$

$$= \frac{1}{2}kQ^2 \left(-\frac{1}{r}\right)_R^\infty = \frac{1}{2}k\frac{Q^2}{R}$$

$$= \frac{1}{2}QV(R) \checkmark$$

$E = 10^6\,V/m$ $\qquad\qquad$ $R = 5\,cm$

$$E = \frac{kQ}{R^2}$$

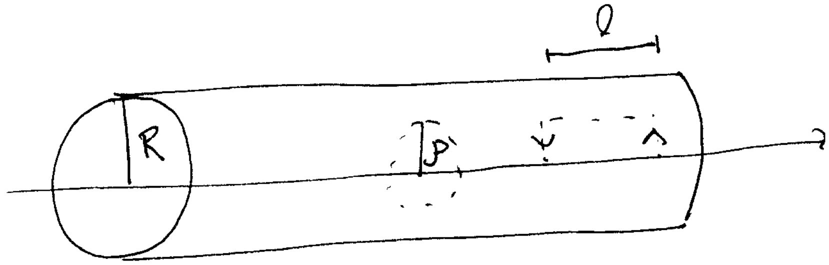
$$Q = \frac{ER^2}{k}$$

$$U = \frac{1}{2} Q V(R) = \frac{1}{2}\left(\frac{ER^2}{k}\right)\left(\frac{kQ}{R}\right)$$

$$= \frac{1}{2} EQR = \frac{1}{2} ER\left(\frac{ER^2}{k}\right)$$

$$= \frac{1}{2}\frac{E^2R^3}{k} = 2\pi\varepsilon_0 E^2R^3$$

6.10



For $\rho < R$, $\quad J = \dfrac{I}{\pi R^2}$

$$I_{enc} = \pi \rho^2 J = \frac{I\rho^2}{R^2}$$

$$B = \frac{\mu_0 I_{enc}}{2\pi\rho} = \frac{\mu_0 I \rho^2/R^2}{2\pi \rho} =$$

$$\vec{B} = \frac{\mu_0 I \rho}{2\pi R^2}\,\hat{\phi}$$

Outside, $\quad \vec{B} = \dfrac{\mu_0 I}{2\pi\rho}\,\hat{\phi}$

Assume $\quad A(0) = 0$

Use loop above

$$\oint \vec{A}\cdot d\vec{l} = \Phi = l\int_0^\rho d\rho\, B = l\int_0^\rho d\rho\, \frac{\mu_0 I \rho}{2\pi R^2}$$

$$A\ell = \frac{\mu_0 I \ell}{2\pi R^2} \quad \frac{\rho^2}{2}$$

$$\vec{A}(\rho) = \left(\frac{\mu_0 I}{4\pi R^2} \rho^2 \right) \hat{z}$$

Outside the wire,

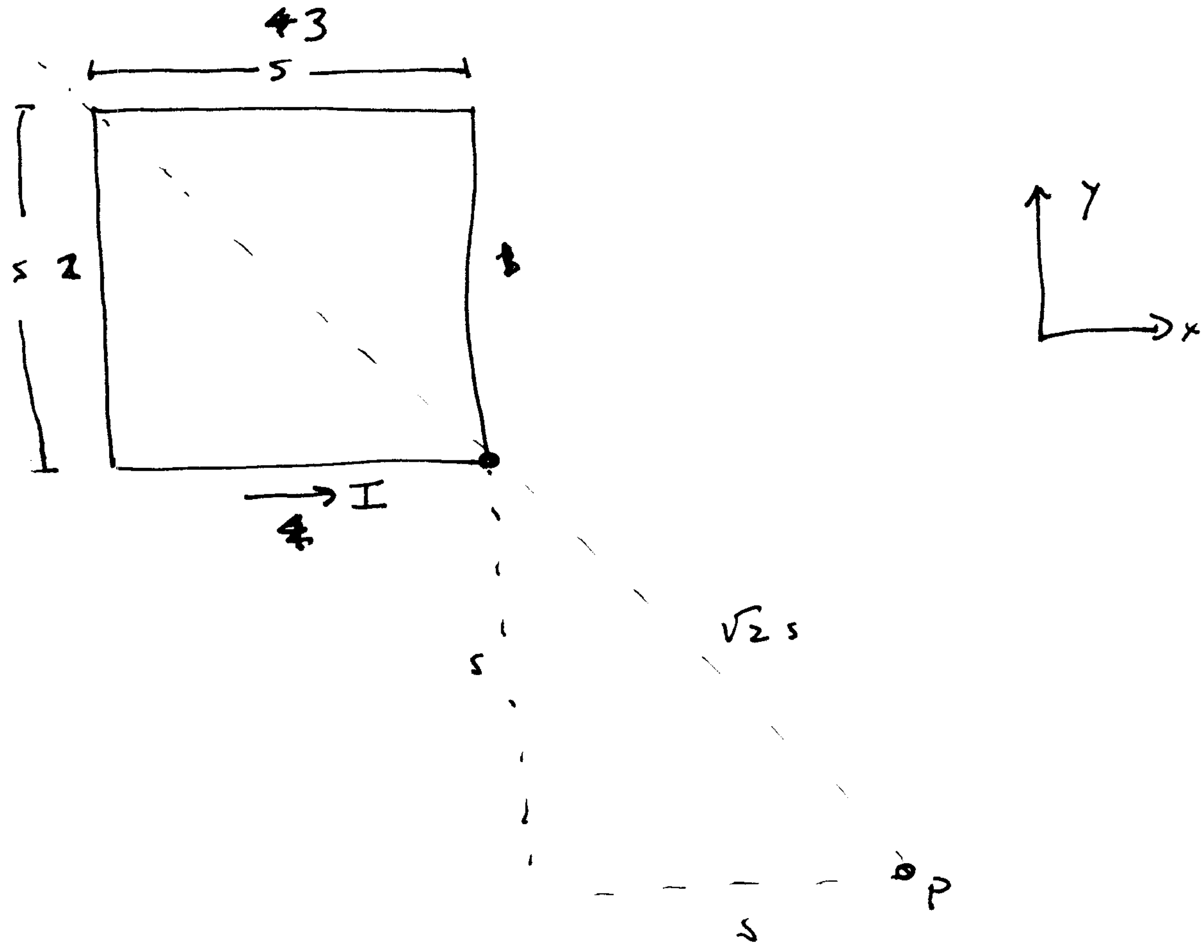
$$\Phi_m = \ell \int_0^R d\rho \, B_i(\rho) + \ell \int_R^\rho d\rho \, B_o(\rho)$$

$$= \frac{\mu_0 I \ell}{4\pi} + \frac{\ell \mu_0 I}{2\pi} \int_R^\rho \frac{d\rho}{\rho}$$

$$= \frac{\mu_0 I \ell}{4\pi} + \frac{\mu_0 I \ell}{2\pi} \ln\left(\rho/R\right)$$

$$A\ell = \Phi \qquad \vec{A}_o(\rho) = \frac{\mu_0 I}{4\pi}\left(1 + 2 \ln \rho/R\right)$$

6.20



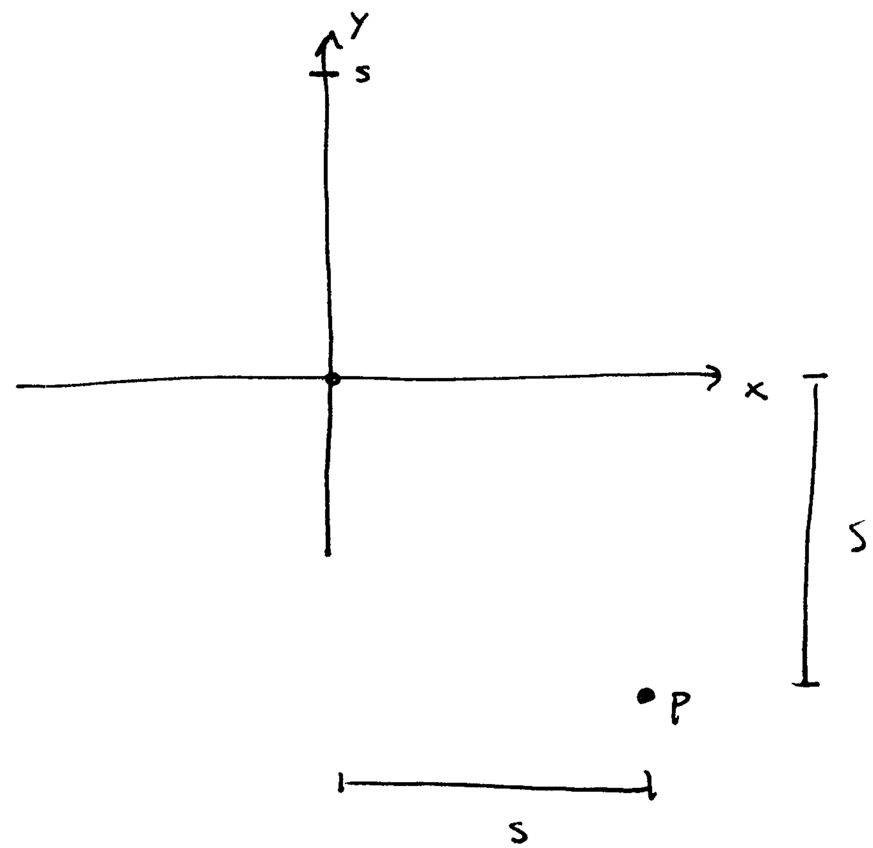
$$\vec{A} = \vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4$$

$\vec{A}_1$ and $\vec{A}_4$ are equidistant from P, so the field is

$$\vec{A}_1 + \vec{A}_4 = A_1\hat{y} + A_1\hat{x}$$

Likewise, $\vec{A}_2 + \vec{A}_3 = -A_2\hat{x} - A_2\hat{y}$

Work on segment 1



The field point at $\vec{r} = (-s, -s, 0)$

The current is at $\vec{r}\,' = (0, y', 0)$

$$r'' = \sqrt{s^2 + (s+y)^2}$$

$$\vec{A} = A_1 \hat{y}$$

$$A_1 = \frac{\mu_0 I}{4\pi} \int_0^s \frac{dy}{\sqrt{s^2 + (s+y^2)}} \qquad \begin{array}{l} u = s+y \\ du = dy \end{array}$$

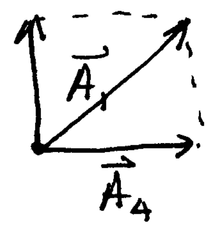
$$= \frac{\mu_0 I}{4\pi} \int_s^{2s} \frac{du}{\sqrt{s^2 + u^2}}$$

$$A_1 = \frac{\mu_0 I}{4\pi} \ln\left(u + \sqrt{u^2+s^2}\right)\Bigg|_{s}^{2s}$$

$$= \frac{\mu_0 I}{4\pi}\left[\ln\left(2s+\sqrt{5s^2}\right) - \ln\left(s+\sqrt{2s^2}\right)\right]$$

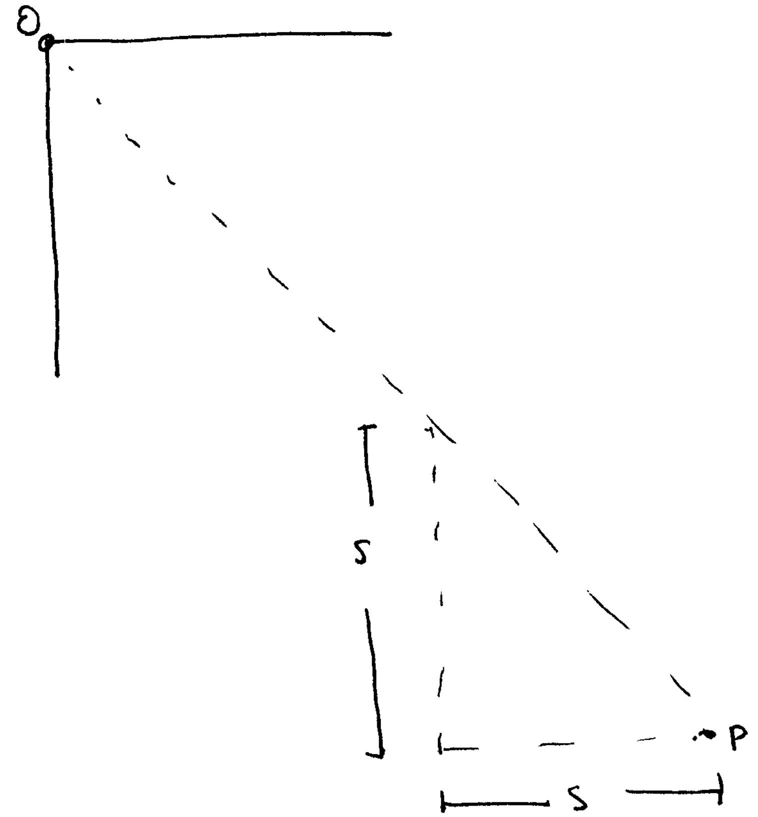
$$A_1 = \frac{\mu_0 I}{4\pi} \ln\left(\frac{2+\sqrt{5}}{1+\sqrt{2}}\right)$$

At point P,



So $\cancel{\vec{A}_1+\vec{A}_4}$ Now let's add componentwise, get the other fields.

2 + 3 are equidistant from P. Let the origin be the corner.



$$\vec{r}_p = (-2s, 2s, 0)$$

$$\vec{r}\,' = (x, 0, 0)$$

$$\vec{r}\,'' = (-2s, 2s - x', 0)$$

$$r'' = \sqrt{4s^2 + (2s - x')^2}$$

$$|\vec{A}_3| = \frac{\mu_0 I}{4\pi} \int_0^s \frac{dx'}{\sqrt{4s^2 + (2s - x')^2}}$$

$u = 2s - x'$

$du = -dx'$

$$|A_3| = -\frac{\mu_0 I}{4\pi}\int_{2s}^{s}\frac{du}{\sqrt{4s^2+u^2}}$$

$$= -\frac{\mu_0 I}{4\pi}\left(\ln\left(u+\sqrt{u^2+4s^2}\right)\right)\Big|_{2s}^{s}$$

$$= \frac{\mu_0 I}{4\pi}\ln\left(\frac{2s+\sqrt{8s^2}}{s+\sqrt{5s^2}}\right)$$

$$= \frac{\mu_0 I}{4\pi}\ln\left(\frac{2+2\sqrt{2}}{1+\sqrt{5}}\right)$$

$$\vec{A} = A_1\hat{x} + A_1\hat{y} - A_3\hat{x} - A_3\hat{y}$$

$$= (A_1 - A_3)\hat{x} + (A_1 - A_3)\hat{y}$$

$$A_1 - A_3 = \frac{\mu_0 I}{4\pi}\left(\ln\left(\frac{2+\sqrt{5}}{1+\sqrt{2}}\right) - \ln\left(\frac{2+2\sqrt{2}}{1+\sqrt{5}}\right)\right) \equiv A_0$$

$$\vec{A} = A_0\hat{x} + A_0\hat{y}$$

(6.22) $\rho = \dfrac{Q}{\frac{4}{3}\pi R^3}$ $\quad Q = 79e$

$A = 197$

$Z = 79$

$$= \frac{79e}{\frac{4}{3}\pi\left((1.2\times10^{-15}m)(197)^{1/3}\right)^3}$$

$$= 1.44\times10^{-15}\ C/m^3$$

Fields

Outside

$$\vec{E} = \frac{kQ}{r^2}\hat{r}$$

$$V = \frac{kQ}{r}$$

Inside

$$\vec{E} = \frac{\rho r}{3\varepsilon_0}\hat{r}$$

$$V = -\int E\,dr$$

$$= -\frac{\rho r^2}{6\varepsilon_0} + V(R)$$

where $V(R)$ is the constant required to make V continuous.

$$V = \frac{kQ}{R} = -\frac{\rho R^2}{6\varepsilon_0} + V(R)$$

but we don't actually need this, we just need the potential difference $\Delta V_{RO} = \dfrac{\rho R^2}{6\varepsilon_0}$

so $V(0) = \Delta V_{RO} + \dfrac{kQ}{R}$

$$= \frac{\rho R^2}{6\varepsilon_0} + \frac{Q}{4\pi\varepsilon_0 R}$$

$$= \frac{\left(Q/\tfrac{4}{3}\pi R^3\right)R^2}{6\varepsilon_0} + \frac{Q}{4\pi\varepsilon_0 R}$$

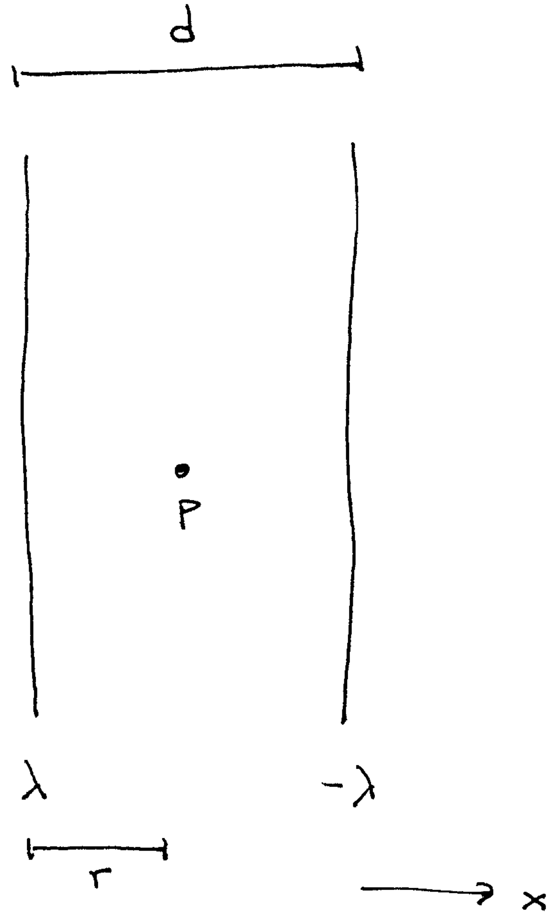
$$= \frac{3Q}{24\pi\varepsilon_0 R} + \frac{Q}{4\pi\varepsilon_0 R}$$

$$= \frac{Q}{4\pi\varepsilon_0 R}\left(1 + \frac{1}{2}\right) = \frac{3}{2}\left(\frac{Q}{4\pi\varepsilon_0 R}\right)$$

$$= 2.4 \times 10^7 \, V$$

6.26 By symmetry $\vec{A} = 0$

6.28



The field is

$$\vec{E} = \left(\frac{\lambda}{2\pi\varepsilon_0 r} + \frac{\lambda}{2\pi\varepsilon_0 (d-r)} \right) \hat{x}$$

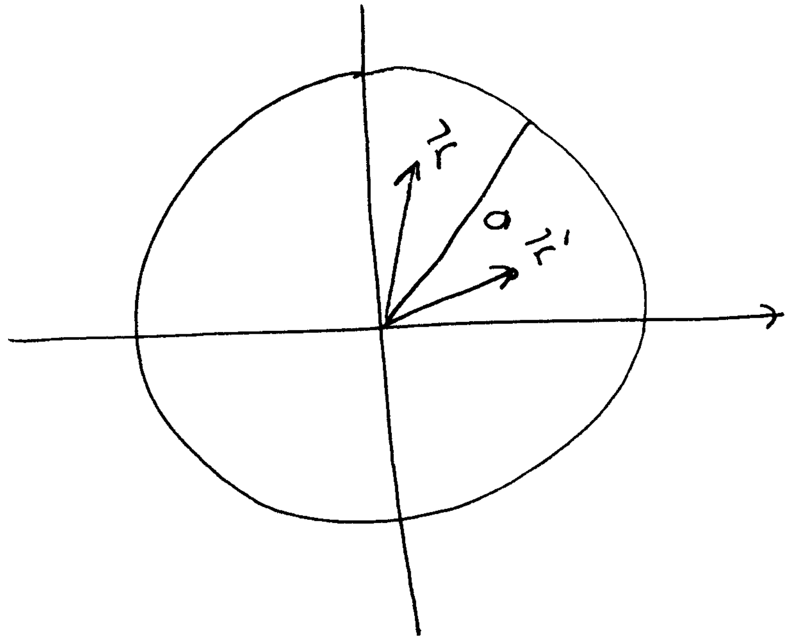
$$V(r) = -\int E\,dr$$

$$= \frac{-\lambda}{2\pi\varepsilon_0} \left(\ln(r) - \ln(d-r) \right) + C$$

The distance from the positive wire is $r_+ = r$ and the distance from the negative wire is $r_- = d-r$

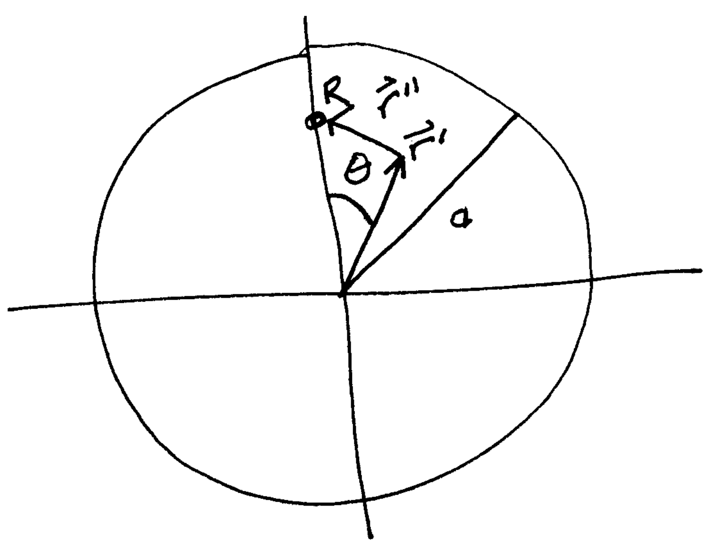
$$V(r) = \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{r_-}{r_+} + C$$

6.36



$$V(\vec{r}) = k \int \frac{da'}{r''}$$

By symmetry, V only depends on R the distance from the origin. So calculate $V(R)$ at a convienient point



$$r''^2 = R^2 + \rho^2 - 2R\rho\cos\theta$$

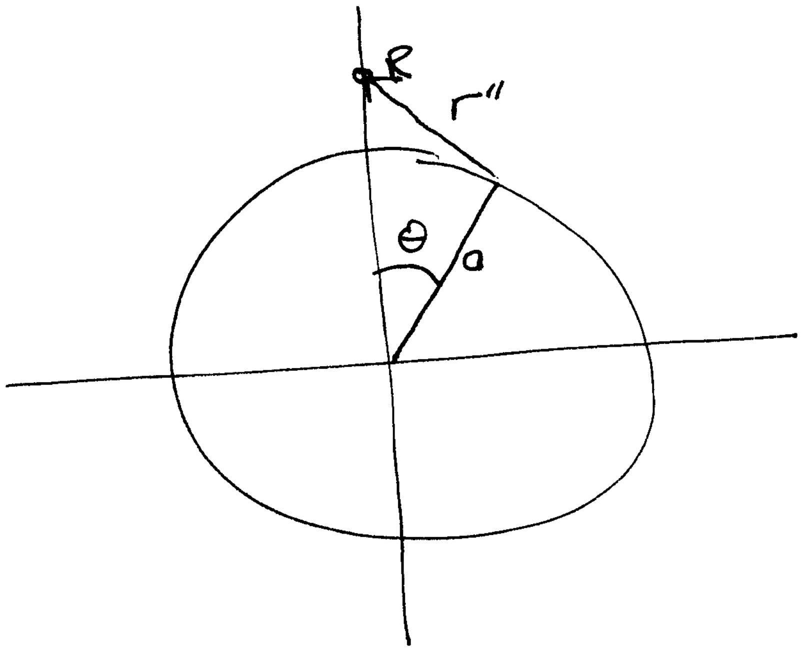
$$V(R) = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} d\theta \int_0^a \rho\, d\rho\, \frac{\sigma}{r''}$$

$$= \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi} d\theta \int_0^a d\rho\, \frac{\rho}{\sqrt{R^2+\rho^2-2R\rho\cos\theta}}$$

This is a mess. If θ integral is performed first you get a fantastic mess. Let's try another strategy.

$$\cancel{V(R) = \frac{\sigma}{4\pi\varepsilon_0}\, 2\rho \sqrt{\frac{(R-\rho)^2}{(R+\rho)^2}}}$$

Compute field of a ring of charge λ



$$r''^2 = R^2 + a^2 - 2Ra\cos\theta$$

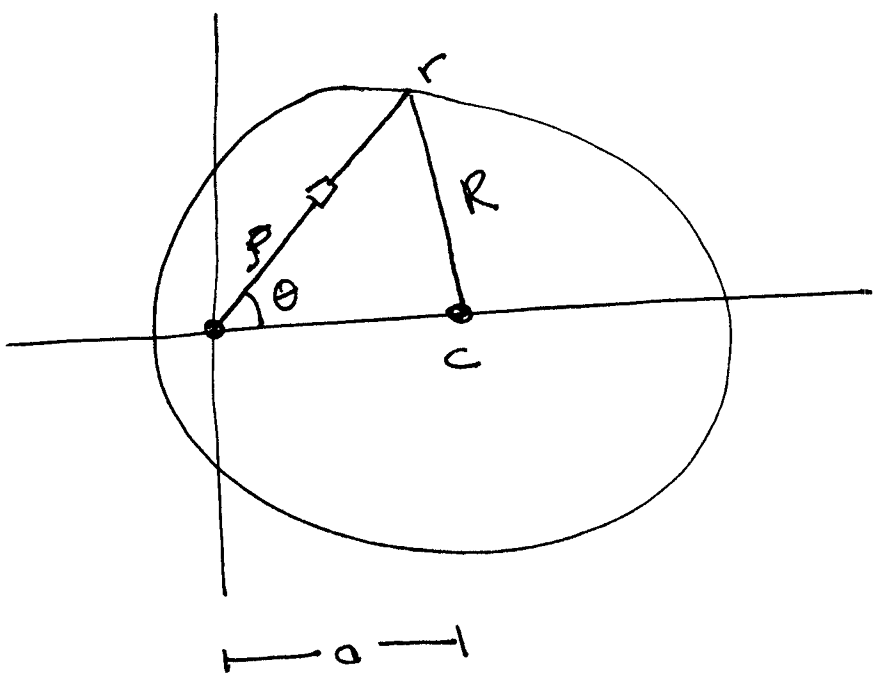
$$V(R) = 2\lambda \int_0^{\pi} \frac{R\,d\theta}{\sqrt{R^2+a^2-2Ra\cos\theta}} =$$

elliptical Function

$$\frac{RK4\lambda \sqrt{\dfrac{-2Ra+R^2+a^2}{(a+R)^2}}\; K\left(2\sqrt{\dfrac{Ra}{(a+R)^2}}\right)}{\sqrt{(a-R)^2}}$$

which is also terrible.

Try original problem with different origin,



Let r be the distance from the origin to the edge of the disk; r is a function of θ.

$$V = 2k\sigma \int_0^{\pi} d\theta \underbrace{\int_0^{r} \frac{\rho\, d\rho}{\rho}}_{=r}$$

$$= 2k\sigma \int_0^{\pi} r\, d\theta$$

Law of cosines, $R^2 = r^2 + a^2 - 2ar\cos\theta$

Solve for r   Quadratic eqn

$$r^2 - 2ar\cos\theta + (a^2 - R^2) = 0$$

$$\frac{2a\cos\theta \pm \sqrt{4a^2\cos^2\theta + 4(R^2 - a^2)}}{2}$$

Choose + root $(r > 0)$

$$r = a\cos\theta + \sqrt{a^2\cos^2\theta + (R^2 - a^2)}$$

$$V = 2k\sigma \int_0^{\pi} \left(a\cos\theta + \sqrt{a^2\cos^2\theta + (R^2 - a^2)} \right) d\theta$$

$$\cancel{V = 4k\sigma\sqrt{R^2 - a^2}\; E(m)}$$

$m =$

$$V = 2k\sigma \int_0^{\pi} \underbrace{a\cos\theta}_{0} + \sqrt{R^2 - a^2\sin^2\theta}\; d\theta$$

$$= 2k\sigma R \int_0^{\pi} \sqrt{1 - \frac{a^2}{R^2}\sin^2\theta}\; d\theta$$

$$= 4k\sigma R \int_0^{\pi/2} \sqrt{1 - m\sin^2\theta}\; d\theta \qquad m = \frac{a^2}{R^2}$$

$$= 4k\sigma R\, E(m)$$

complete elliptic integral
second kind

7.4

$$\Delta U = mB\cos\theta - mB\cos\pi$$

$$= 2mB$$

$$= 2\,(3\times10^{-4}\ \mathrm{A\,m^2})(0.6\,\mathrm{T})$$

$$= 3.6\times10^{-4}\,\mathrm{J}$$

7.6 $\quad \sigma = \sigma_0 \cos\theta$

$Q = 0$

$Q_z = 2(p_{zz} - p_{xx})$

$\cancel{P_{xx} = \sigma_0 \int x'^2 \cos\theta' R\,d\theta'}$

$p_{xx} = \sigma_0 \int x'^2 \cos\theta'\,da' = \sigma_0 \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' R^2 \sin\theta' \cos\theta' x'^2$

$p_{zz} = \sigma_0 \int z'^2 \cos\theta'\,da' = \sigma_0 \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' R^2 \sin\theta' \cos\theta' z'^2$

$x' = R\sin\theta' \cos\phi' \qquad z' = R\cos\theta'$

$p_{xx} = 0$ because $\int d\phi' \cos\phi' = 0$

$p_{zz} = \sigma_0 R^3 2\pi \int_0^{\pi} \sin\theta' \cos'\theta^3\,d\theta'$

$u = \cos\theta \qquad \cancel{v = \cos^4\theta}$

$du = -\sin\theta\,d\theta \qquad \cancel{dv = -3\cos^3\theta' \sin\theta'}$

$$P_{zz} = -\sigma_0 R^3 2\pi \int_1^{-1} u^3 du$$

$$= -\sigma_0 R^3 2\pi \left. \frac{u^4}{4} \right|_{-1}^{1} = 0$$

So the quadrapole moment is zero.

Dipole Moment $\qquad P_x = P_y = 0$

$$P_z = \sigma_0 \int_0^{\pi} d\theta' \int_0^{2\pi} R^2 \sin\theta' d\phi' \cos\theta' z'$$

$$= \sigma_0 R^3 \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \sin\theta' \cos^2\theta' d\theta'$$

using $z' = R\cos\theta'$

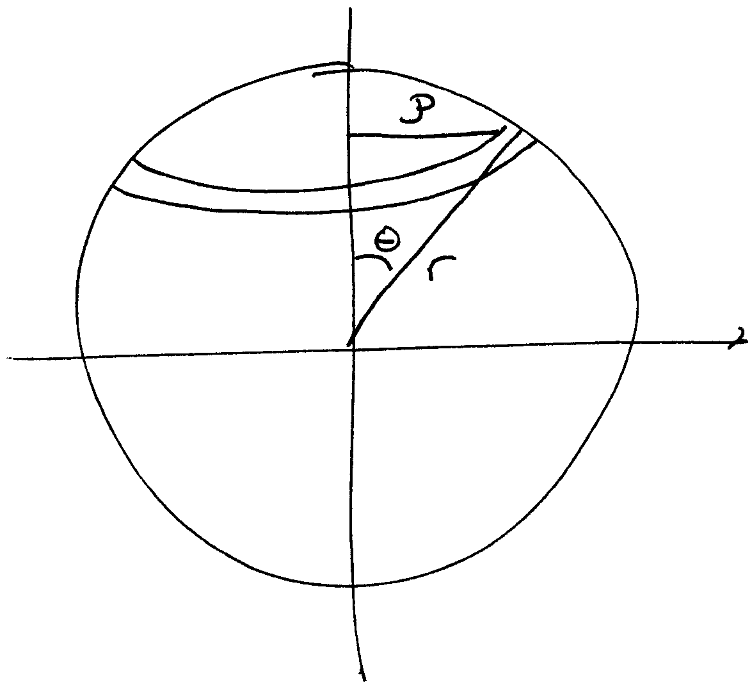
$$P_z = 2\pi\sigma_0 R^3 \int_0^{\pi} d\theta' \sin\theta' \cos^2\theta'$$

$u = \cos\theta' \qquad du = -\sin\theta' d\theta'$

$$P_z = -2\pi\sigma_0 R^3 \int_{\cancel{0}}^{\cancel{\pi}} u^2 du = -2\pi\sigma_0 R_3 \left. \frac{u^3}{3} \right|_{1}^{-1}$$

$$= \frac{4}{3}\pi R^3_{\cancel{3}} \sigma_0$$

7.10



The current is $\quad dI = \underbrace{(\omega \rho)}_{v} \underbrace{(\sigma R \, d\theta)}_{\lambda}$

$$\rho = R \sin\theta$$

$$dI = \sigma \omega R^2 \sin\theta \, d\theta$$

The magnetic moment is

$$dm = dI \cdot \text{area} = dI \, (R \sin\theta)^2 \pi$$

$$= \sigma \pi \omega R^4 \sin^3\theta \, d\theta$$

$$m = \int_0^\pi dm = \sigma \pi \omega R^4 \int_0^\pi \sin^3\theta \, d\theta = \frac{4}{3} \pi R^4 \sigma \omega$$

in the z. direction.