(1) Since we are given the polarization, compute the charges and apply Gauss.

<u>Volume charge</u>

$$\rho_b = -\nabla \cdot \vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \gamma r$$

$$= -3\gamma$$

<u>Surface Charge</u>

$$\sigma_b = \vec{P}\cdot\hat{n} = \gamma a$$

<u>Total Charge</u>

$$Q_T = \frac{4}{3}\pi a^3 \rho_b + 4\pi a^2 \sigma_b$$

$$= -4\pi a^3 \gamma + 4\pi a^3 \gamma = 0$$

$$\Rightarrow \vec{E}_o = 0 \qquad \vec{D}_o = \varepsilon_0 \vec{E}_o = 0$$

Apply Gauss' Law to a spherical surface of radius r inside the material

$$\Phi_e = 4\pi r^2 E_i = \frac{Q_{enc}}{\varepsilon_0}$$

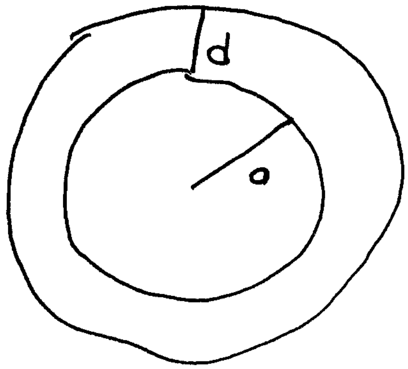
$$Q_{enc} = \left(\frac{4}{3}\pi r^3\right) - 3\gamma = -4\pi\gamma r^3$$

$$\vec{E}_i = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{-4\pi\gamma r^3}{4\pi\varepsilon_0 r^2}\hat{r} = -\frac{\gamma r}{\varepsilon_0}\hat{r}$$

$$\vec{D}_i = \varepsilon_0 \vec{E}_i + \vec{P} = \varepsilon_0\left(\frac{-\gamma r}{\varepsilon_0}\right)\hat{r} + \gamma\hat{r}$$

$$= 0 \checkmark$$

2



$$\varepsilon_r = \varepsilon_a + (\varepsilon_b - \varepsilon_a)\frac{(r-a)}{d}$$

The sphere is held at V_0, which establishes a charge Q_f on the sphere. This is the only free charge so

$$\nabla \cdot \vec{D} = \rho_f$$

$$\int \vec{D} \cdot d\vec{a} = Q_f$$

Use spherical Gaussian $r > a$

$$4\pi r^2 D = Q_f$$

$$\vec{D} = \frac{Q_f}{4\pi r^2}\hat{r}$$

Outside the dielectric

$$\vec{D} = \varepsilon_0 \vec{E}_o$$

$$\vec{E}_o = \frac{Q_f}{4\pi\varepsilon_0 r^2}\hat{r}$$

Inside the dielectric

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}_i$$

$$\vec{E}_i = \frac{Q_f}{4\pi\varepsilon_0 \varepsilon_r r^2}\hat{r}$$

$$= \frac{Q_f}{4\pi\varepsilon_0} \frac{1}{\varepsilon_a + \varepsilon_b - \varepsilon_a (\cancel{r}-a)/d}\hat{r}$$

<u>Potential Difference</u>

$$\Delta V_{\infty, d+a} = \frac{Q_f}{4\pi\varepsilon_0 (a+d)} \quad \text{by observation}$$

$$\Delta V_{d+a, a} = -\int \vec{E} \cdot d\vec{l} \quad > 0$$

$$= -\int_a^{a+d} \frac{Q_f}{4\pi\varepsilon_0 \varepsilon_r} dr \qquad d\vec{l} = -\hat{r}\,dr$$

>

> _EnvAllSolutions := true;

$$\_EnvAllSolutions := true \tag{1}$$

> $int\left(\frac{1}{\left(ea+\frac{(eb-ea)\cdot(r-a)}{d}\right)\cdot r^2},\ r=a..(a+d)\right)$ *assuming* $a>0$ *and* $d>0$;

$$\left\{[undefined,\ ,\ \mathrm{And}\left(a<\Re\left(\frac{ea\,d-eb\,a+ea\,a}{-eb+ea}\right),\Re\left(\frac{ea\,d-eb\,a+ea\,a}{-eb+ea}\right)<a+d\right)],\ [\right. \tag{2}$$

$$-\Bigg(\mathrm{piecewise}\Bigg(a=\frac{ea\,d-eb\,a+ea\,a}{-eb+ea},\ \mathrm{signum}\left(\frac{-eb+ea}{ea^2\,d^2-2\,ea\,d\,eb\,a+2\,ea^2\,d\,a+eb^2\,a^2-2\,eb\,a^2\,ea+ea^2\,a^2}\right)\infty,\ -(d\,(-\ln(a)\,ea\,a-eb\,a\ln(d)-eb\,a\ln(-ea)+ea\,d-eb\,a+ea\,a\ln(d)+ea\,a\ln(-ea)+\ln(a)\,eb\,a+ea\,a))/\left(a\left(ea^2\,d^2-2\,ea\,d\,eb\,a+2\,ea^2\,d\,a+eb^2\,a^2-2\,eb\,a^2\,ea+ea^2\,a^2\right)\right)\Bigg)\Bigg)+$$

$$\Bigg(\mathrm{piecewise}\Bigg(a+d=\frac{ea\,d-eb\,a+ea\,a}{-eb+ea},\ \mathrm{signum}\left(\frac{-eb+ea}{ea^2\,d^2-2\,ea\,d\,eb\,a+2\,ea^2\,d\,a+eb^2\,a^2-2\,eb\,a^2\,ea+ea^2\,a^2}\right)\infty,\ -(d\,(ea\,d-eb\,a+ea\,a-d\,eb\ln(d)-d\,eb\ln(-eb)+\ln(a+d)\,eb\,a-\ln(a+d)\,ea\,a+d\ln(a+d)\,eb-d\ln(a+d)\,ea+ea\,d\ln(d)+ea\,d\ln(-eb)-eb\,a\ln(d)-eb\,a\ln(-eb)+ea\,a\ln(d)+ea\,a\ln(-eb)))/\left(3\,a\,d^2\,ea^2+d^3\,ea^2-4\,d\,eb\,ea\,a^2-2\,d^2\,eb\,ea\,a+3\,d\,ea^2\,a^2+eb^2\,a^3+eb^2\,a^2\,d-2\,eb\,a^3\,ea+ea^2\,a^3\right)\Bigg)\Bigg),\ ,\ otherwise]$$

>

$$\Delta V_{d+a,a} = +\frac{Q_f}{4\pi\varepsilon_0}\int_a^{a+d}\frac{dr}{\left(\varepsilon_a + (\varepsilon_b-\varepsilon_a)\frac{(r-a)}{d}\right)r^2}$$

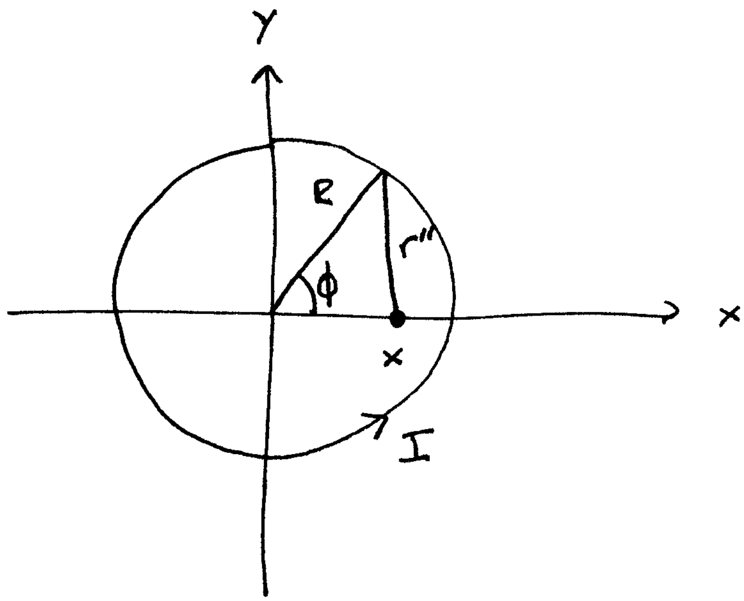
At this point, the problem becomes too difficult for a test question.

Total Potential Difference

$$\Delta V_{\infty a} = \frac{Q_f}{4\pi\varepsilon_0(d+a)} + \Delta V_{d+a,a}$$

$$C = \frac{Q_f}{\Delta V}$$

(3)



Compute the field at a point x along the x-axis.

Symmetry will give you the rest.

$$\vec{A} = \frac{\mu_0}{4\pi}\int_0^{2\pi} \frac{d\vec{l}\, I}{r''} \qquad d\vec{l} = \hat{\phi}\, R\, d\phi$$

$$r''^2 = R^2 + x^2 - 2Rx\cos\phi$$

$$\vec{A} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{\hat{\phi}\, R\, d\phi}{\sqrt{R^2+x^2-2Rx\cos\phi}}$$

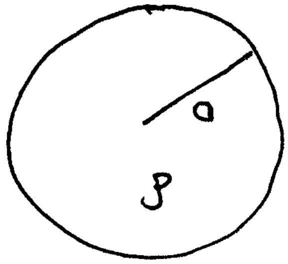
By symmetry, the x component of $\vec{A}$ is zero.

Convert $\hat{\phi}$ into cartesian, from Griffiths

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\vec{A} = \frac{\mu_0 I \hat{y} R}{4\pi} \int_0^{2\pi} \frac{\cos\phi \, d\phi}{\sqrt{R^2 + x^2 - 2Rx\cos\phi}}$$

4 4



The energy outside the sphere is the same as the energy of a spherical capacitor of radius a,

$$U_o = \frac{1}{2} Q_t V$$

$$= \frac{1}{2} \frac{Q_t^2}{4\pi\varepsilon_0 a}$$

$$Q_t = \text{Total Charge} = \frac{4}{3}\pi a^3 \rho$$

Inside the sphere the field is

$$\vec{E}_i = \frac{\rho r}{3\varepsilon_0} \hat{r}$$

$$U_i = \int \frac{1}{2}\varepsilon_0 E^2 dv = \frac{1}{2}\varepsilon_0 \int_0^a 4\pi r^2 E^2 \, dr$$

$$= \left(\frac{1}{2}\varepsilon_0\right)(4\pi)\left(\frac{\rho}{3\varepsilon_0}\right)^2 \int_0^a r^4 dr$$

$$U_i = \frac{2\pi\rho^2}{9\varepsilon_0}\;\frac{a^5}{5}$$

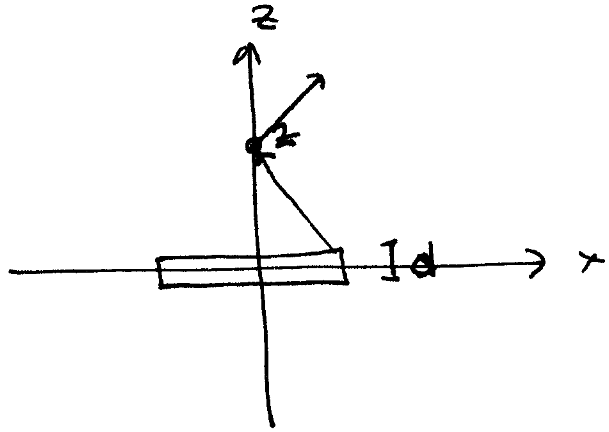
$$\rho a^3 = \frac{3}{4\pi} Q_f$$

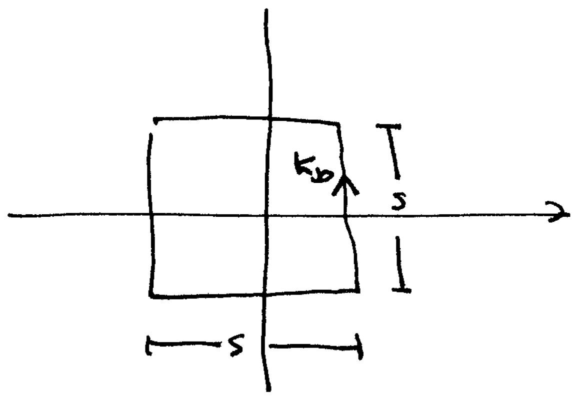
$$U_i = \frac{2\pi}{45\varepsilon_0 a}\left(\frac{3}{4\pi}\right)^2 Q_f^2$$

$$U_i = \frac{Q_f^2}{(8\pi)(5\varepsilon_0)\,a} = \frac{1}{10}\;\frac{Q_f^2}{(4\pi\varepsilon_0 a)}$$

$$U_T = \left(\frac{1}{2} + \frac{1}{10}\right)\frac{Q_f^2}{4\pi\varepsilon_0 a} = \frac{3}{5}\;\frac{Q_f^2}{4\pi\varepsilon_0 a}$$

(5)





The magnetization produces a surface current $\vec{K}_b$ around the magnet. $\vec{K}_b = \vec{M} \times \hat{n}$

$$K_b = M_0 \text{ counterclockwise}$$

This produces a total current

$$I = K_b d$$

The field at a point $z = 1\text{cm}$ is four times the field of one side. You guys have calculated the field of a square current before, so I will leave it to you.

(6) At the distance 10cm >> 1cm, a dipole approximation should be good.

The total moment is $\vec{m} = \vec{M} V = (10^{6}\,\mathrm{A/m})(10^{-3}\,\mathrm{m})(10^{-2}\,\mathrm{m})^2$

$$= 10\,\mathrm{A\,m^2}$$

The field of a dipole along its axis is sufficiently important that it should be on your formula sheet

$$\vec{B}(0,0,z) = \frac{\mu_0}{4\pi}\frac{2m}{|z|^2}\hat{z}$$

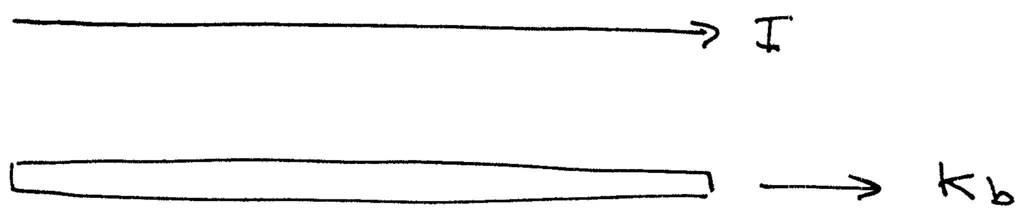
or can be derived from the vector potential.

$$\vec{B}(0,0,10\,\mathrm{cm}) = \left(\frac{4\pi\times10^{-7}\,\mathrm{Tm/A}}{4\pi}\right)\frac{2\,(10\,\mathrm{Am^2})}{(0.1\,\mathrm{m})^3}\hat{z}$$

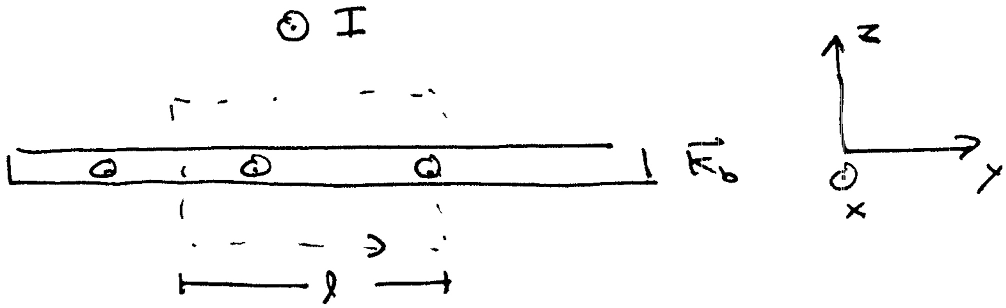
$$= \cancel{2\times10^{-4}\,\mathrm{T}\,\hat{z}}$$

$$= 2\times10^{-3}\,\mathrm{T}\,\hat{z}$$

7



End View



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

$$-B_t l + B_b l = \mu_0 K_b l$$

$$\vec{B}_t = -\frac{\mu_0 K_b}{2}\hat{y}$$

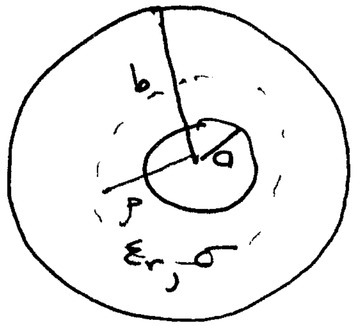
The force on the wire is (Length l)

$$\vec{F} = I\vec{L}\times\vec{B} = I(l\hat{x})\times\left(-\frac{\mu_0 K_b}{2}\hat{y}\right)$$

$$= -\frac{I l \mu_0 K_b}{2}\hat{z}$$

$$\vec{F}/l = -\frac{I\mu_0 K_b}{2}\hat{z}$$

8



If a charge per unit length λ is placed on the inner conductor, the displacement field is

$$\int \vec{D} \cdot d\vec{a} = Q_{enc}$$

$$2\pi \rho l D = \lambda l$$

using a cylindrical Gaussian surface of radius ρ

$$\vec{D} = \frac{\lambda}{2\pi\rho} = \varepsilon_0 \varepsilon_r \vec{E}$$

$$\varepsilon_r = 1 + \chi_e$$

$$\vec{E} = \frac{\lambda}{2\pi \varepsilon_0 \varepsilon_r \rho} \hat{\rho}$$

The potential difference between the conductors is

$$\Delta V = -\int_a^b \vec{E}\cdot d\vec{l} = -\frac{\lambda}{2\pi\varepsilon_0\varepsilon_r}\ln\left(\frac{b}{a}\right) \equiv V_0$$

The current is

$$\vec{J} = \sigma\vec{E} = \frac{\lambda\sigma}{2\pi\varepsilon_0\varepsilon_r\rho}$$

The total current is (for a distance ρ from center)

$$I = 2\pi\rho\, l\, J(\rho)$$

$$= \frac{\lambda\sigma}{\varepsilon_0\varepsilon_r}$$

or in terms of V_0

$$\lambda = +\frac{2\pi\varepsilon_0\varepsilon_r V_0}{\ln(b/a)}$$

(Yes I'm killing - signs because of an undefined sign of potential.

$$I = \frac{\sigma V_0 2\pi}{\ln(b/a)}$$

The ε_r cancelled, because the V_0 boundary condition sets the electric field.

The resistance would be

$$\frac{R}{g} = \frac{V_0}{I} = \frac{\ln(b/a)}{2\pi\sigma}$$

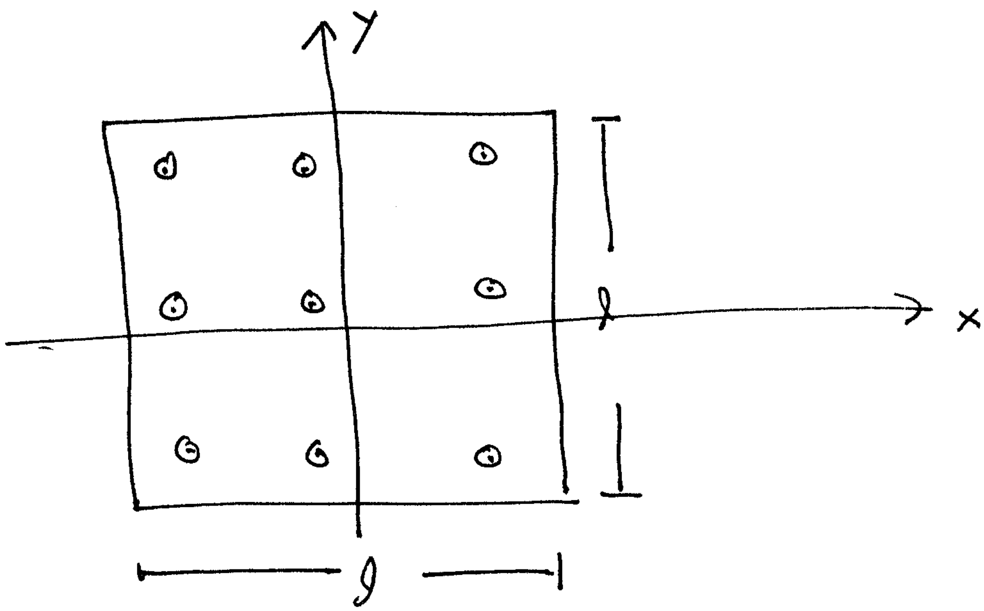
(9) The energy stored is found most easily from the capacitance. We did most of the work in (8)

$$C = \frac{Q}{\Delta V} = \frac{\lambda l}{\Delta V}$$

$$\frac{C}{l} = \frac{2\pi \varepsilon_0 \varepsilon_r V_0}{V_0 \ln(b/a)} = \frac{2\pi \varepsilon_0 \varepsilon_r}{\ln(b/a)}$$

$$\frac{U}{l} = \frac{1}{2}\frac{C}{l} V_0^2 = \frac{\pi \varepsilon_0 \varepsilon_r}{\ln(b/a)} V_0^2$$

(10)



$$R = \frac{\rho A}{4l} \qquad \Phi_m = l^2 B_o \cos\omega t$$

$$\text{emf} = -\frac{d\phi}{dt} = l^2 B_o \omega \sin\omega t$$

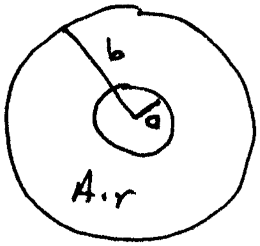
$$I = \frac{\text{emf}}{R} = \frac{l^2 B_o \omega}{R} \sin\omega t$$

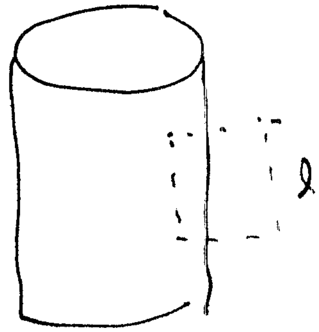
The flux is decreasing from 0 to $\pi/2\omega$

so the current is counterclockwise to oppose the

change in flux.

11





$$\int \vec{H} \cdot d\vec{l} = N'Il$$

$$H_i l = N'Il$$

$H_i = N'I$ for the airspace in the solenoid

$$B_i = \mu_0 N'I$$

and in the iron

$$B_r = \mu_0 \mu_r N'I$$

The total flux through the solenoid is

$$\Phi_m = N B_i \left(\pi b^2 - \pi a^2\right) + N B_r \pi a^2$$

$$= N N' I \mu_0 \left(\pi b^2 + (\mu_r - 1)\pi a^2\right)$$

The inductance is

$$L = \frac{\Phi_m}{I} = N N' \mu_0 \left(\pi b^2 + (\mu_r - 1) \pi a^2 \right)$$

or the inductance per unit length

$$\frac{L}{l} = (N')^2 \mu_0 \left(\pi b^2 + (\mu_r - 1) \pi a^2 \right)$$