PHYS 3414 - Electricity and Magnetism- Homework Set 11

Practice for Test 3. Some of these problems are too difficult for a test, but parts of them would make good test questions.

1 A point charge, q, is a distance D from the center of a grounded conducting sphere of radius R. Compute the potential at a distance $D/2$ on a line between the point charge and the center of the sphere.

2 A capacitor is formed of two plates that are not quite parallel. One plate is in the $x-y$ plane and the other plate is tipped at an angle α from the $x-y$ plane. The nearest point between the two plates has separation d. The plates are square and have area of the plates is $A = \ell^2$ and they are filled with a dielectric with relative permittivity ϵ_r. Compute the capacitance ignoring fringing.

3 A sphere of radius a is held at potential V_0. Compute the potential and field at a distance $4a$ from the origin.

4 A sphere of radius a is held at a potential $V = V_0(\cos\theta + 1)$. Compute the field inside the sphere.

5 A dielectric cylinder with radius a and relative permittivity ϵ_r is placed in a uniform field such that the field far from the cylinder is $\vec{E} = E_0\hat{x}$. Compute the field everywhere.

6 A thin infinite cylinder with radius a is covered with a surface charge density σ. The cylinder is parallel to an infinite grounded conducting plane and is a distance D from the plane. Compute the force per unit length on the cylinder.

7 A channel, infinite in the z direction, has potential on the surface $V(x,0) = 0$, $V(a,y) = 0$, $V(x,b) = 0$, and $V(0,y) = V_0\sin(2\pi y/b)$. Compute the field at the center of the channel.

8 A cylindrical system with radius a has potential $V_i = A\rho^4\cos(4\phi)$ inside and $V_o = B\rho^{-4}\cos(4\phi)$ outside. Compute the charge density on the surface.

9 An infinite cylinder of radius a is held at potential V_0 for $0 < \phi < \pi/2$ and zero for $\pi/2 < \phi < 2\pi$. Compute the potential outside the cylinder.

10 A sphere of radius a is held at potential V_0 for $0 < \theta < \pi/4$ and zero for $\pi/4 < \theta < \pi$. Compute the first two non-zero terms in the expansion of the potential outside the sphere.

1

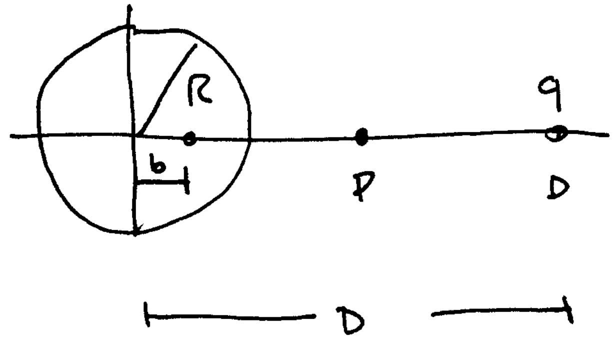


Image charge $q' = -\frac{R}{D} q$

$$V(D/2) = \frac{kq}{(D - D/2)} + \frac{kq'}{D/2 - b}$$

Image Charge Location

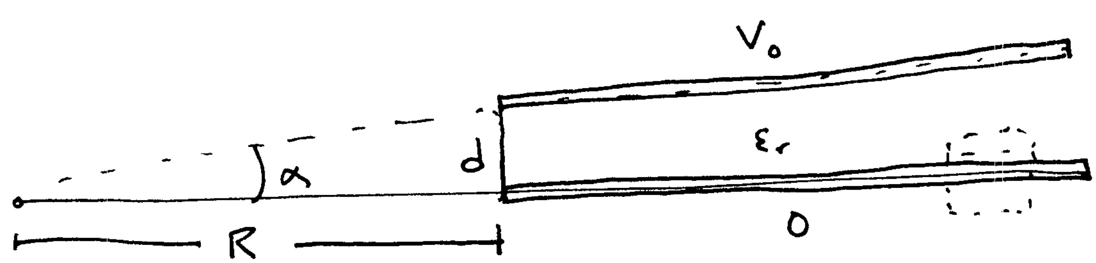
$$b = \frac{R^2}{D}$$

$$V(D/2) = \frac{2kq}{D} - \frac{kq(R/D)}{\frac{D}{2} - \frac{R^2}{D}}$$

$$V\left(\frac{D}{2}\right) = \frac{2kq}{D} - \frac{kqR}{\frac{D^2}{2} - R^2}$$

$$= \frac{kq}{D}\left(2 - \frac{R}{\frac{D}{2} - \frac{R^2}{D}}\right)$$

(2) Capacitor with tipped plates. Work in cylindrical coordinates



Let the plate in the x-y plane be at $V=0$
and the upper plate be at $V=V_0$

Solve without the dielectric.

There is no charge between the plates, so use
Laplace's eqn in cylindrical coordinates.

The trivial solution $V = A + B\phi$ works
with $A=0$ and $B = \frac{V_0}{\alpha}$

$$V(\rho,\phi) = \frac{V_0\,\phi}{\alpha}$$

The electric field between the plates is

$$\vec{E} = -\nabla V = -\frac{V_0}{\alpha \rho}\hat{\phi}$$

The charge ~~on the~~ density on the bottom plate, using a Gaussian pillbox.

$$\Phi = \vec{E}_{conductor} \cdot \hat{n} A - \vec{E} \cdot \hat{n} A = \frac{Q_{enc}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

$$0 - \frac{V_0}{\alpha \rho} = \frac{\sigma}{\epsilon_0}$$

$$\sigma = -\frac{\epsilon_0 V_0}{\alpha \rho}$$

Compute the total charge on the bottom plate.

$$Q = l \int_R^{R+l} \sigma \, d\rho$$

$$= -\frac{V_0 l}{\alpha} \int_R^{R+l} \frac{d\rho}{\rho} = -\frac{V_0 l}{\alpha} \ln\left(\frac{R+l}{R}\right)$$

$$\frac{d}{R} = \tan\alpha$$

$$R = \frac{d}{\tan\alpha}$$

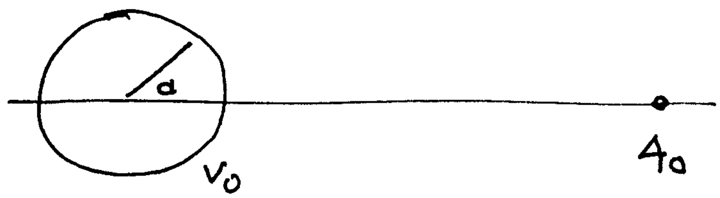
<u>Capacitance</u>

$$C = \frac{|Q|}{V_0} = +\frac{l}{\alpha}\ln\left(\frac{R+l}{R}\right)$$

With the dielectric

$$C_K = \varepsilon_r C_0 = +\frac{\varepsilon_r l}{\alpha}\ln\left(\frac{R+l}{R}\right)$$

3



Method I Laplace's eqn in spherical coordinates.

Use trivial solution $V = \frac{A}{r}$

$$V(a) = \frac{A}{a} = V_0 \qquad A = aV_0$$

$$V = V_0\left(\frac{a}{r}\right)$$

Potential at 4a

$$V(4a) = \cancel{V_0\left(\frac{4a}{a}\right) = 4V_0}$$

$$= V_0\left(\frac{a}{4a}\right) = \frac{V_0}{4}$$

Field

$$\vec{E} = -\nabla V = \frac{V_0 a}{r^2}\,\hat{r}$$

$$\vec{E}(4a) = \frac{V_0}{16a}\,\hat{r}$$

Method II Field is just that of a point charge with Q s.t.

$$\frac{kQ}{a} = V_0$$

$$Q = \frac{V_0 a}{k}$$

$$V = \frac{kQ}{r} = \frac{V_0 a}{r} \qquad V(4a) = \frac{V_0}{4}$$

Field $$\vec{E} = \frac{kQ}{r^2}\hat{r} = \frac{V_0 a}{r^2}\hat{r}$$

$$\vec{E}(4a) = \frac{V_0}{16a}\hat{r}$$

(4) $V(\theta) = V_0 \cos\theta + V_0$

$$= V_0 P_1(\cos\theta) + V_0 P_0(\cos\theta)$$

Inside Sphere the solution to Laplace's eqn

$$V(r,\theta) = \sum_n A_n r^n P_n(\cos\theta)$$

At the surface of the sphere,

$$V(a,\theta) = V_0 P_1(\cos\theta) + V_0 P_0(\cos\theta)$$

$$= \sum A_n a^n P_n(\cos\theta)$$

By observation,

$A_n = 0 \quad n > 1$

$A_0 = V_0$

$A_1 a = V_0$

$A_1 = \frac{V_0}{a}$

Inside

$$V(r,\theta) = V_0 P_0(\cos\theta) + V_0 \frac{r}{a} P_1(\cos\theta)$$

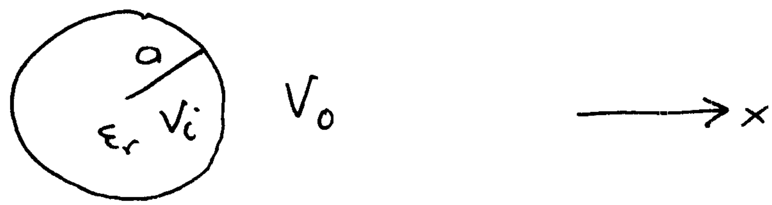
$$= V_0 + V_0 \frac{r}{a} \cos\theta$$

Field Inside

$$\vec{E} = -\nabla V$$

$$= -\frac{V_0}{a}\cos\theta\,\hat{r} + \frac{V_0}{a}\sin\theta\,\hat{\theta}$$

⑤ Cylinder ε_r



<u>Boundary Conditions</u>

- $V_i(a, \phi) = V_o(a, \phi)$ (V continuous)

- Since there is no free charge, the normal component of $\vec{D}$ must be continuous. You can show this from $\nabla \cdot \vec{D} = 0$ and using a pillbox at the surface.

$$|\vec{D}_{normal}| = \varepsilon_r |\vec{E}_{normal}| = \varepsilon_r \frac{\partial V}{\partial r}\bigg|_a$$

$$\varepsilon_r \frac{\partial V_i}{\partial r}\bigg|_a = \frac{\partial V_o}{\partial r}\bigg|_a$$

- At large ρ, $V = -E_0 \rho \cos\phi$

  so $\vec{E} = E_0 \hat{x}$

The potential inside (keeping terms that don't explode) is

$$V_i = \sum_n A_n \rho^n \cos n\phi + B_n \rho^n \sin n\phi$$

The potential outside, keeping terms that don't blow up and the term we need to meet the $\vec{E} = E_0 \hat{x}$ boundary condition.

$$V_o = -E_0 \rho \cos\phi + \sum_n C_n \rho^{-n} \cos n\phi + D_n \rho^{-n} \sin n\phi$$

~~Apply Other Boundary Conditions~~

~~I. V continuous~~ $V_i(a,\phi) = V_o(a,\phi)$

$$V_i = -E_0 a \cos\phi + \sum A_n a^n \cos n\phi + B_n a^n \sin n\phi$$

$$= V_o = \sum_n C_n a^{-n} \cos n\phi + D_n a^{-n} \sin n\phi$$

Boundary Conditions

I. V continuous $\quad V_i(a,\phi) = V_o(a,\phi)$

$$V_i(a,\phi) = \sum_n A_n a^n \cos n\phi + B_n a^n \sin n\phi$$

$$= V_o(a,\phi)$$

$$= -E_o a \cos\phi + \sum_n C_n a^{-n} \cos n\phi + D_n a^{-n} \sin n\phi$$

Use orthogonality

$n \neq 1 \quad A_n a^n = C_n a^{-n}$

$\qquad\quad B_n a^n = D_n a^{-n}$

$n = 1 \quad A_1 a = -E_o a + C_1/a$

$\qquad\quad B_1 a = D_1/a$

II. D_{normal} Continuous

$$\varepsilon_r \frac{\partial V_i}{\partial r}\bigg|_a = \frac{\partial V}{\partial r}\bigg|_a$$

$$\varepsilon_r \frac{\partial V_i}{\partial r}\bigg|_a = \varepsilon_r \sum A_n n a^{n-1} \cos n\phi + B_n n a^{n-1} \sin n\phi$$

$$= \frac{\partial V_o}{\partial r}\bigg|_a = -E_o \cos\phi + \sum_n C_n(-n) a^{-(n+1)} \cos n\phi$$
$$+ D_n(-n) a^{-(n+1)} \sin n\phi$$

Use orthogonality

$n \neq 1$ $\quad \varepsilon_r A_n n a^{n-1} = -n C_n a^{-(n+1)}$

$$\varepsilon_r B_n n a^{n-1} = -n D_n a^{-(n+1)}$$

$n = 1$ $\quad \varepsilon_r A_1 = -E_o - \frac{C_1}{a^2}$

The equations for $n \neq 1$ can be solved if

$$A_n,\ B_n,\ C_n,\ D_n = 0$$

Likewise $\quad B_1 = D_1 = 0$

<u>So we are left to solve</u>

$$\varepsilon_r A_1 = -E_0 - \frac{C_1}{a^2}$$

$$a A_1 = -E_0 a + \frac{C_1}{a}$$

$$A_1 = -E_0 + \frac{C_1}{a^2}$$

$$\varepsilon_r \left(-E_0 + \frac{C_1}{a^2} \right) = -E_0 - \frac{C_1}{a^2}$$

$$(1-\varepsilon_r) E_0 = -\varepsilon_r \frac{C_1}{a^2} - \frac{C_1}{a^2}$$

$$(\varepsilon_r - 1) E_0 = \frac{C_1}{a^2} (\varepsilon_r + 1)$$

$$\boxed{C_1 = \frac{(\varepsilon_r - 1) a^2 E_0}{\varepsilon_r + 1}}$$

$$A_1 = -E_0 + \frac{C_1}{a^2}$$

$$= -E_0 + E_0 \frac{(\varepsilon_r - 1)}{(\varepsilon_r + 1)}$$

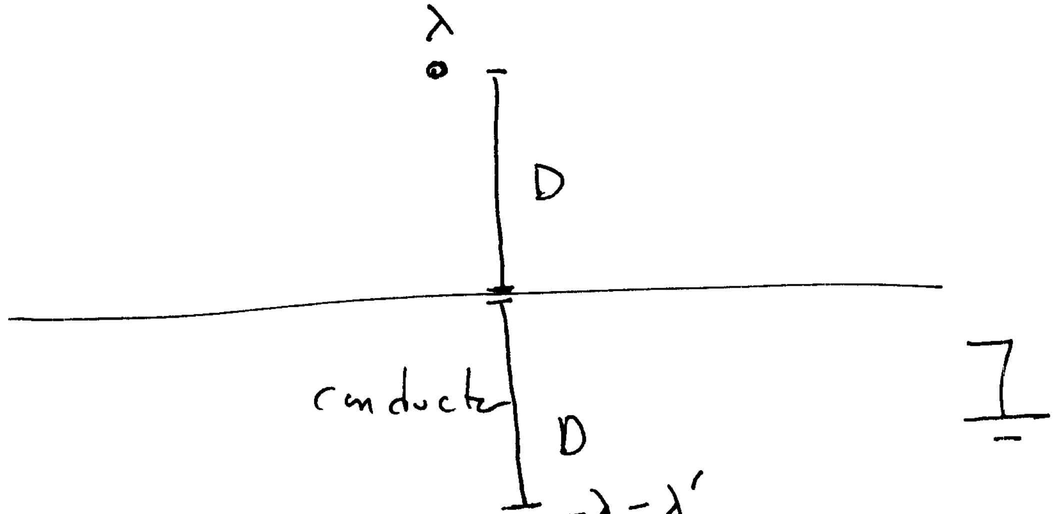
$$= \frac{E_0}{\varepsilon_r + 1}\left(-(\varepsilon_r + 1) + \varepsilon_r - 1 \right.$$

$$= -\frac{2E_0}{\varepsilon_r + 1}$$

The potential is then

$$\boxed{\begin{aligned} V_i &= -\frac{2E_0}{\varepsilon_r + 1} \rho \cos\phi \\ V_o &= -E_0 \rho \cos\phi + \frac{(\varepsilon_r - 1)}{\varepsilon_r + 1} \frac{a^2}{\rho} E_0 \cos\phi \end{aligned}}$$

(6) Thin cylinder, σ = surface charge density, radius a.



<u>Sln</u> Since the cylinder is thin, approximate it by an infinite line charge $\lambda = \sigma \cdot 2\pi a$

Use an image line charge $\lambda' = -\lambda$ a distance D below the plane.

The field of the line charge is

$$E_{line} = \frac{\lambda'}{2\pi\varepsilon_0 r} \qquad r = 2D$$

$$= \frac{\lambda'}{4\pi\varepsilon_0 D}$$

The force per unit length is

$$\vec{F} = \lambda \vec{E}$$

$$|\vec{F}| = \frac{\lambda \lambda'}{4\pi\varepsilon_0 D} = \frac{(\sigma 2\pi a)^2}{4\pi\varepsilon_0 D}$$