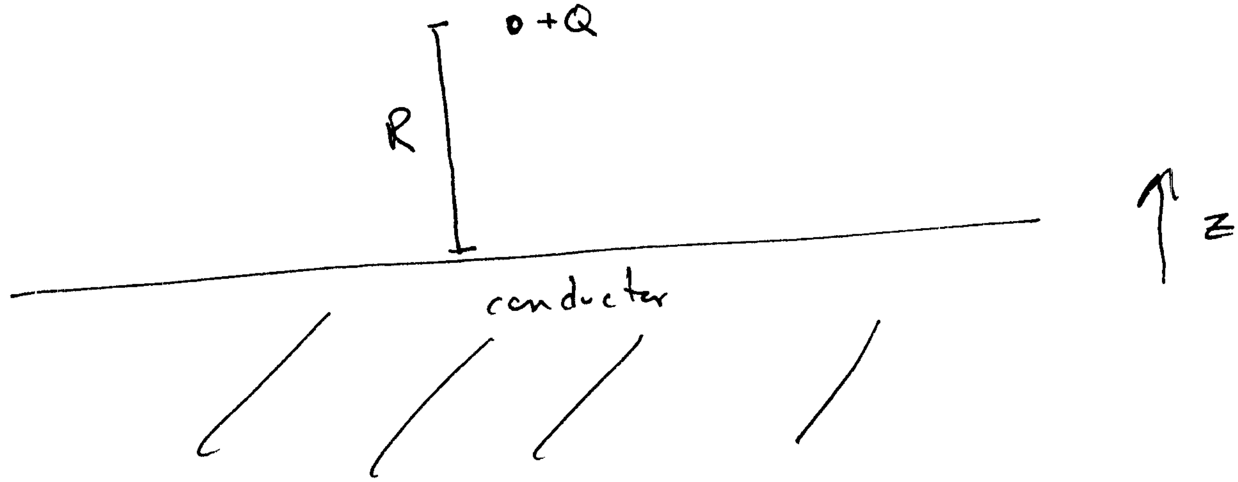
Method of Images

Consider a point charge above an infinite conducting plane. Let the plane be grounded $V=0$.



Find the potential everywhere

Well, we know how to do this. Let the point charge be the center of the system and z be upward.

The system for $z>0$ obeys Poisson's eqn

$$\nabla^2 V = -\rho/\varepsilon_0 = -\frac{Q\,\delta(\vec{r})}{\varepsilon_0}$$

But we only know how to solve Laplace's eqn!

Write the solution as any solution to Poisson's equation for the point charge even if it doesn't fit the boundary conditions V_p and some solution to Laplace's eqn that satisfies the boundary conditions, V_l.

$$V = V_p + V_l$$

So we need some solution of Poisson's eqn for a point charge, $V_p = \frac{KQ}{r}$

Now we need a solution to Laplace's eqn s.t. V satisfies the boundary conditions.

$$V = \frac{KQ}{r} + V_l$$

The system is spherical (sort of) and the potential must go to zero at ∞.

$$V_{\ell} = \sum_n A_n r^{-(n+1)} P_n(\cos\theta)$$

<u>Boundary Conditions</u>

$$V = 0 \quad \text{at} \quad -R\hat{z}$$

Where is the plane?

$$-R = r\cos\theta$$

$$r = -\frac{R}{\cos\theta}$$

$$V(r, \phi, \theta)_{\text{plane}} = 0$$

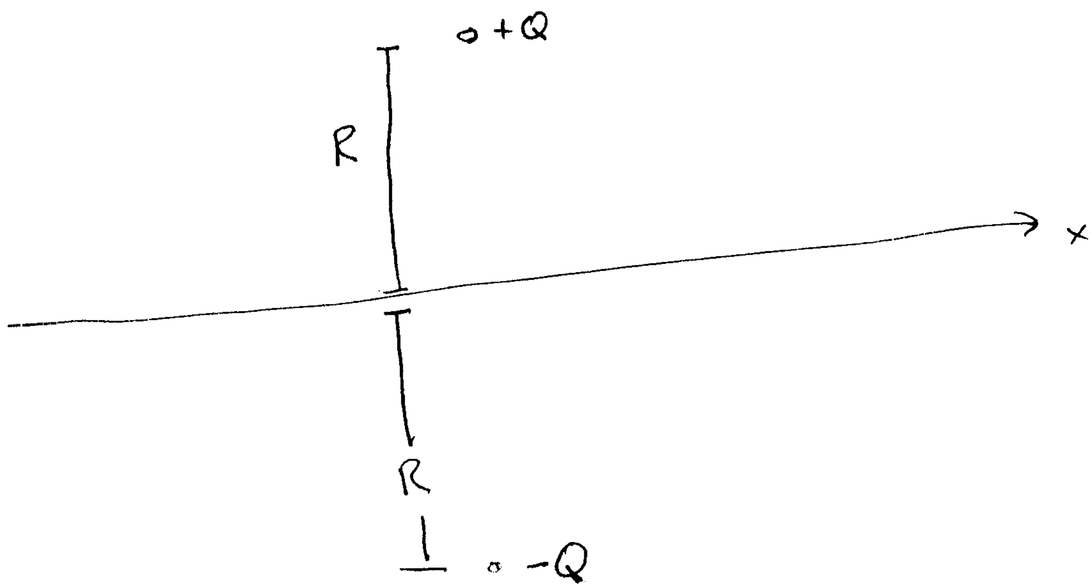
$$V\left(-\frac{R}{\cos\theta}, \phi, \theta\right) = 0$$

$$0 = \sum_n A_n \left(\frac{-R}{\cos\theta}\right)^{-(n+1)} P_n(\cos\theta) + \frac{KQ}{(-R/\cos\theta)}$$

Looks doable but real hard.

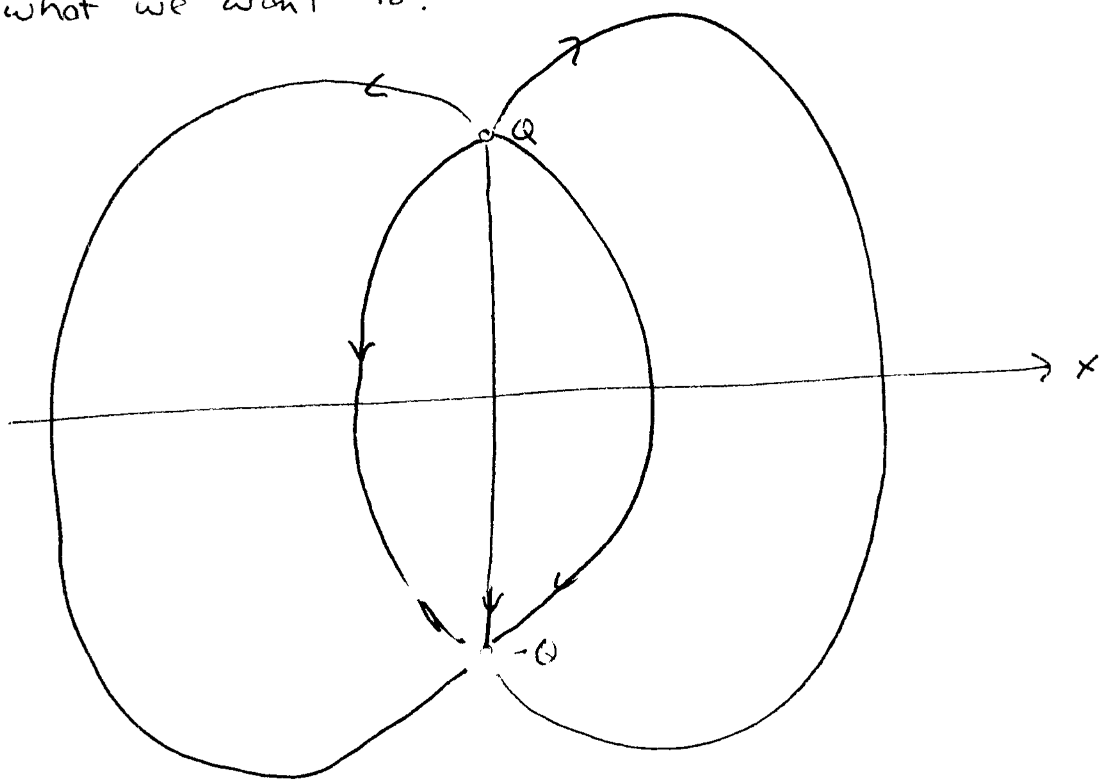
It would be nice to trip over a solution that met the boundary conditions and the use uniqueness to assert it was the only solution.

Consider a dipole system



$$V = \frac{kQ}{r_+} - \frac{kQ}{r_-} = 0$$ on plane since $r_+ = r_-$.

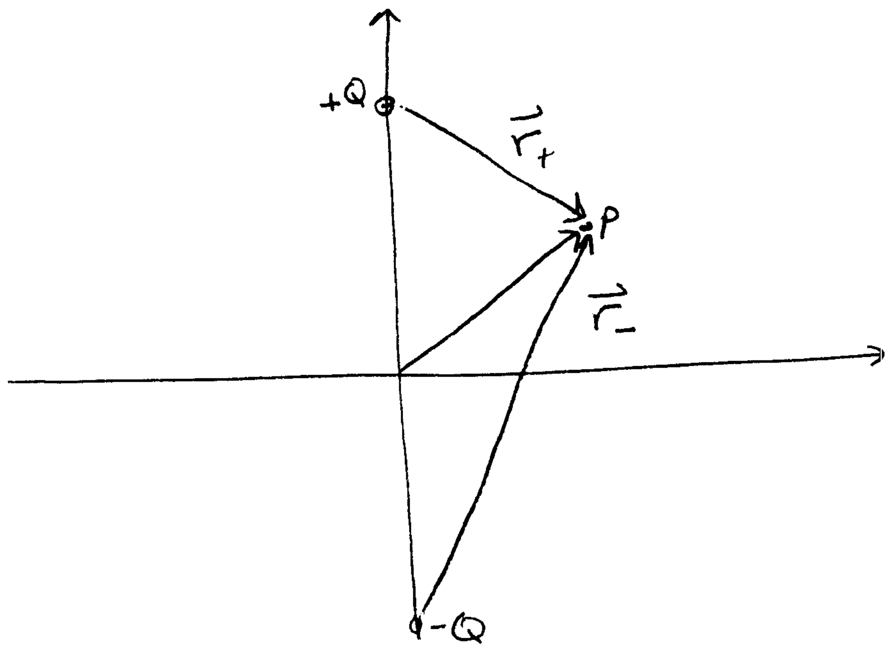
The field of the two charges does exactly what we want to.



$\vec{E} \perp$ x-y plane.

The total charge on the plane is $-Q$ since every line that begins on $+Q$ ends on the plane.

Taking the point on the plane between the charges as the origin,

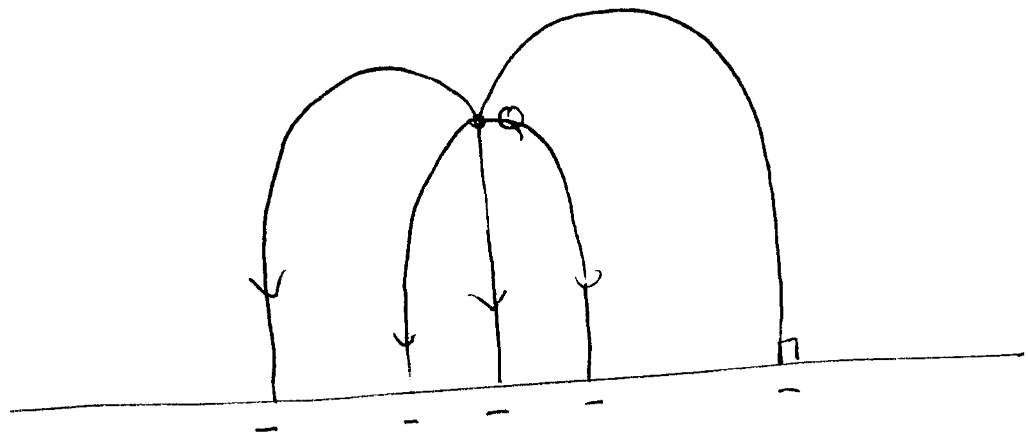


$$V(x,y,z) = \frac{kQ}{\sqrt{x^2+y^2+(R-z)^2}} - \frac{kQ}{\sqrt{x^2+y^2+(R+z)^2}}$$

$$= \frac{kQ}{\sqrt{\rho^2+(R-z)^2}} - \frac{kQ}{\sqrt{\rho^2+(R+z)^2}}$$

Since the potential satisfies the boundary conditions it is the correct potential ~~at~~ for $z>0$. The charge $-Q$ is called the <u>image charge</u> and this method of solution, the method of images.

Anything we can calculate for the two charge system is correct for the charge plane system for $z>0$.



For example, the force exerted by the plane on Q

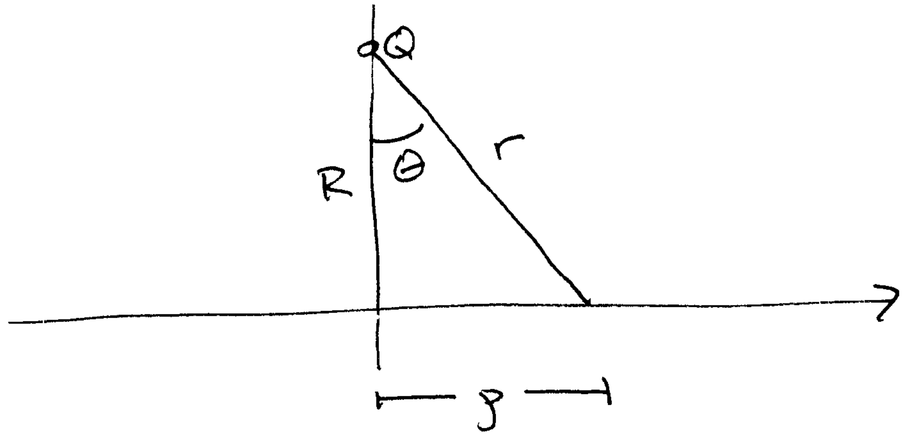
is $$F = -\frac{kQ^2}{(2R)^2}\,\hat{z}$$

the force of the image charge.

The field at the plane is $$\vec{E}_{plane} = \vec{E}_+ + \vec{E}_-$$

$$= 2E_+ \cos\theta\,\hat{z}$$

$$= \frac{-2kQ\cos\theta}{(\rho^2+R^2)}\,\hat{z}$$



$$\cos\theta = \frac{R}{r} = \frac{R}{\sqrt{\rho^2+R^2}}$$

$$\vec{E} = \frac{-2KQR}{(\rho^2+R^2)^{3/2}}\,\hat{z}$$

The charge on the plane is $\sigma = \varepsilon_0\,\hat{z}\cdot\vec{E}$

$$= \frac{-2K\varepsilon_0 QR}{(\rho^2+R^2)^{3/2}} = \frac{-QR}{2\pi(\rho^2+R^2)^{3/2}}$$

The energy of the system is

$$U = \frac{1}{2}\left(\frac{kQ^2}{2R}\right)$$

the 1/2 because half the energy is stored in fields $z<0$.