Images II

We can also use the method of images to solve problems with grounded conducting spheres, cylinders, and dielectric materials.

Grounded Conducting Sphere

Radius R, point charge q a distance a from center.

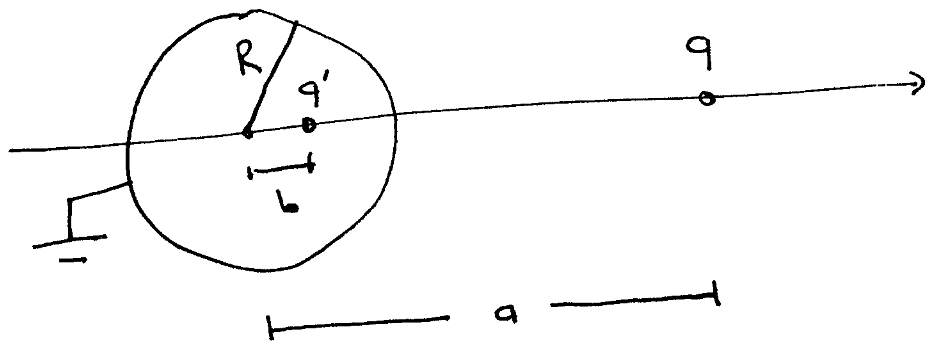


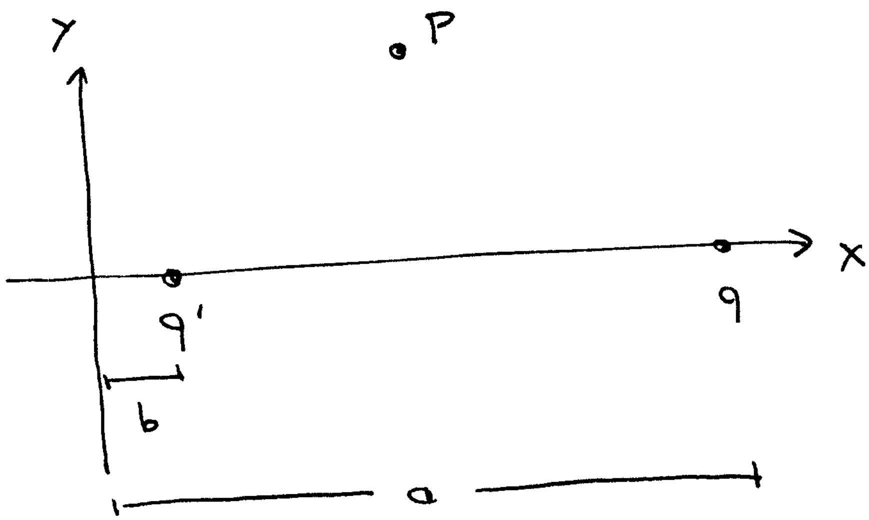
Image Charge $q' = -\frac{R}{a} q$

at $b = \frac{R^2}{a}$

Total Charge on Sphere

– All field lines beginning at q end at q' (on the surface of the sphere). So the sphere has charge q'.

The potential outside the sphere



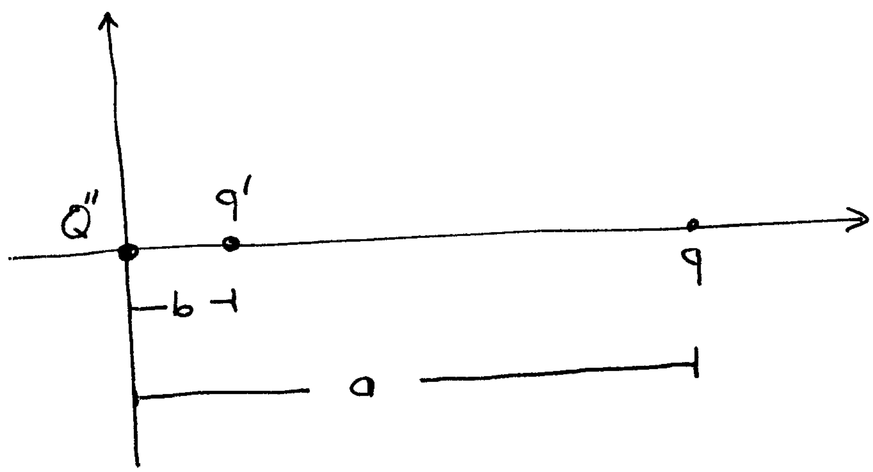
$$V(x,y,z) = \frac{kq}{\sqrt{(x-a)^2+y^2+z^2}} + \frac{kq'}{\sqrt{(x-b)^2+y^2+z^2}}$$

Other cases

(1) Charged conducting sphere with charge Q

⇒ The same image charge brings the surface of the sphere to zero potential but leaves it with ~~q~~ $q'<0$ charge.

⇒ If a charge $Q'' = Q - q'$ is placed on the surface, or equivalently at the origin, it will spread out uniformly. The potential for $r>R$ will be the same for the system.



(2) Conducting sphere at potential V_0. Now we add a charge at the origin that raises the surface of the sphere to V_0.

$$V(R) = \frac{KQ'''}{R} = V_0$$

$$Q''' = V_0 R / K$$

Conducting Cylinders

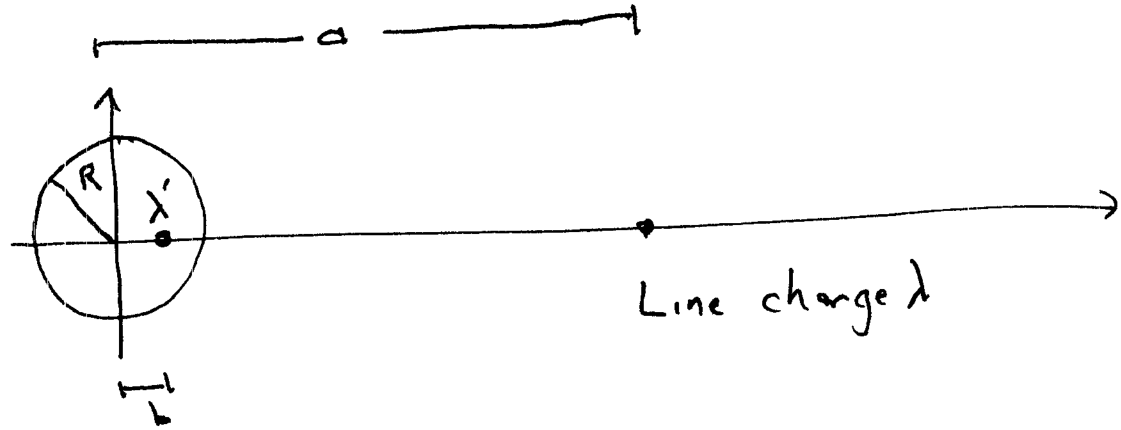
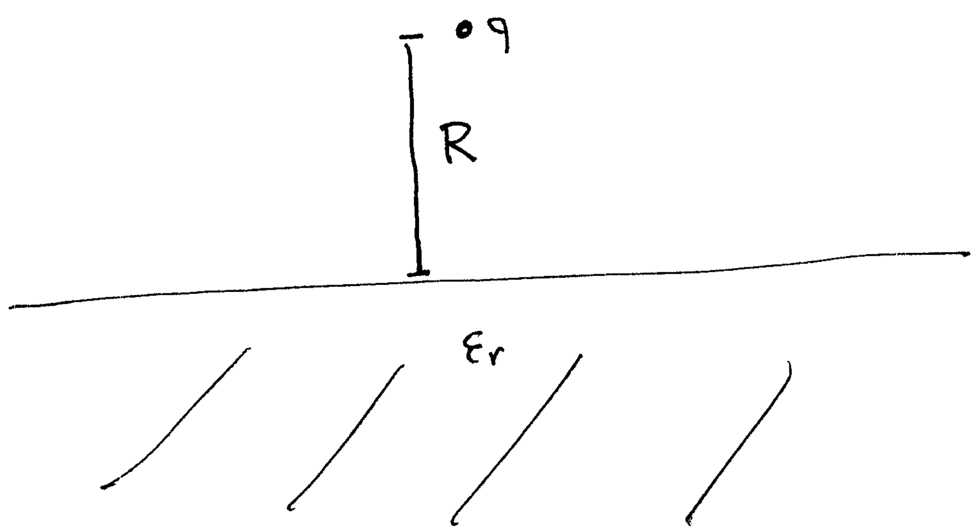


Image Line Charge $\lambda' = -\lambda$

at $b = \dfrac{R^2}{a}$

Images of Dielectric Surfaces

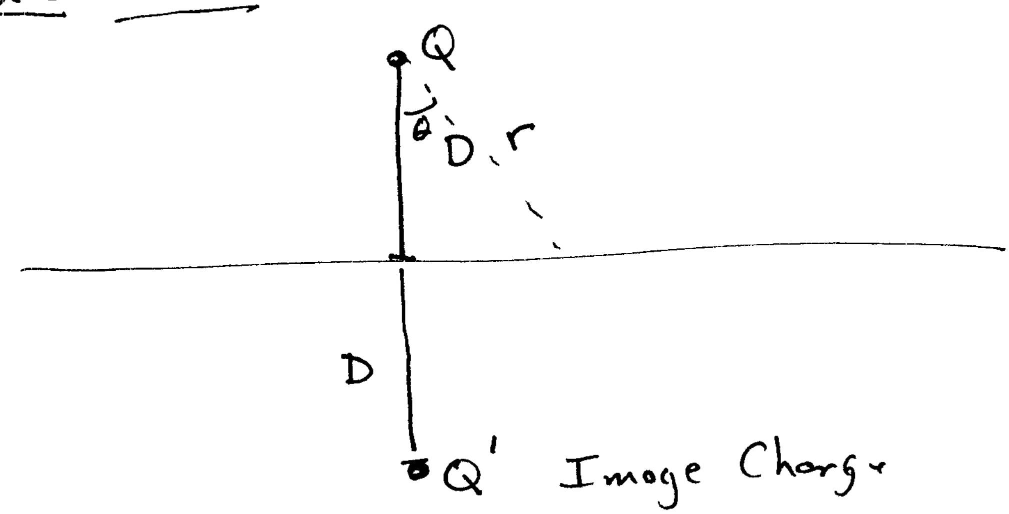


Two cases

(1) Field outside dielectric

(2) Field inside dielectric

Case I Outside



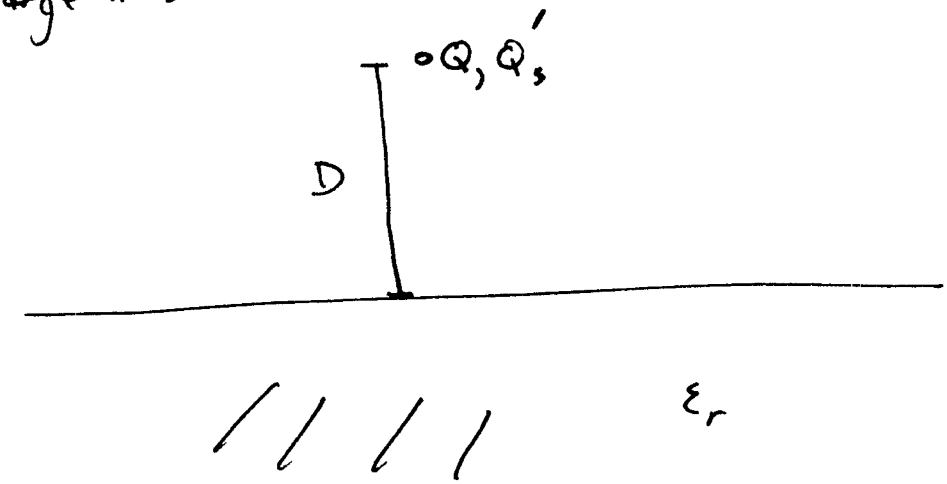
Place image charge equally spaced a distance D from the interface.

$$Q' = -\frac{\varepsilon_r - 1}{\varepsilon_r + 1} Q$$

This charge combination yields a surface charge density

$$\sigma_b = -\frac{(\varepsilon_r - 1) Q \cos\theta}{2\pi(\varepsilon_r + 1) r^2}$$

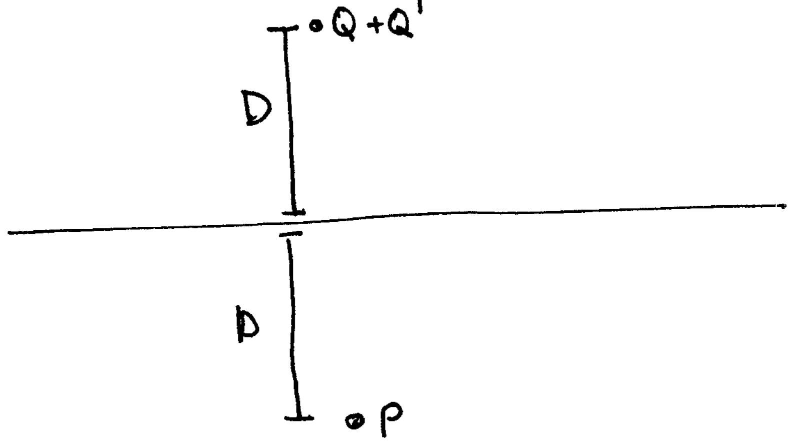
<u>Case II</u> Field inside dielectric - Now we can't place the image charge in the dielectric. The image charge must be in air.



This time place image charge at same place as real charge to form a total charge $Q'+Q$

<u>Ex</u> A point charge Q is a distance D from a linear dielectric with dielectric constant $\varepsilon_r = 2$. Compute the field at a distance D in the dielectric, and the Polarization at this point. Compute the force the dielectric exerts on Q.

To compute the field in the dielectric we need to place the image charge outside of the dielectric.



The image charge we need is

$$Q' = -\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1}\right) Q = -\left(\frac{2-1}{2+1}\right) Q$$

$$= -\frac{2}{3} Q$$

The field at P a distance 2D from Q+Q' is

$$\vec{E} = -\frac{k(Q+Q')}{(2D)^2}\,\hat{z} = -\frac{k\,{}^1\!/_3\,Q}{4D^2}\,\hat{z}$$

$$= -\frac{kQ}{12D^2}\,\hat{z}$$

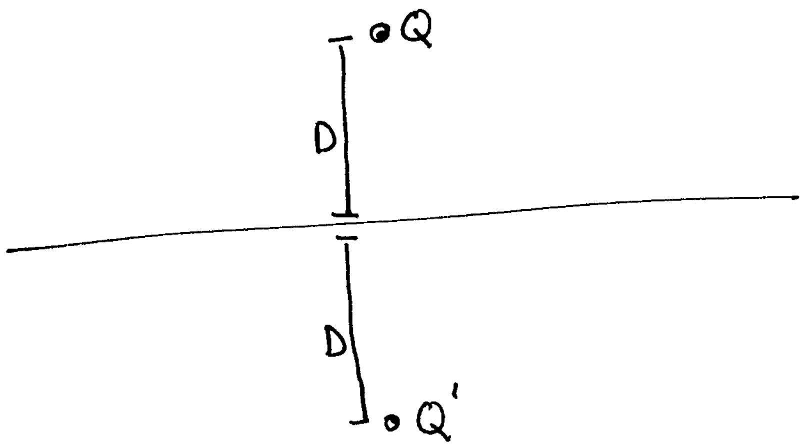
$\Longrightarrow$ $\frac{1}{3}$ the field you would have had without dielectric.

The polarization can be found from the field

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 (\varepsilon_r - 1) \vec{E} = \cancel{k}\, \varepsilon_0 \vec{E}$$

$$= -\frac{\varepsilon_0 k Q}{12 D^2} \hat{z} = \frac{-Q}{48 \pi D^2} \hat{z}$$

Force on the point charge – Now place Q' in the dielectric.



$$\vec{F} = \frac{k Q Q'}{(2D)^2} \hat{z} = \frac{k Q \left(-\frac{2}{3} Q\right)}{(2D)^2} \hat{z}$$

$$= -\frac{k Q^2}{6 D^2} \hat{z}$$

(conductor would have given $-\frac{kQ^2}{4D^2} \hat{z}$)

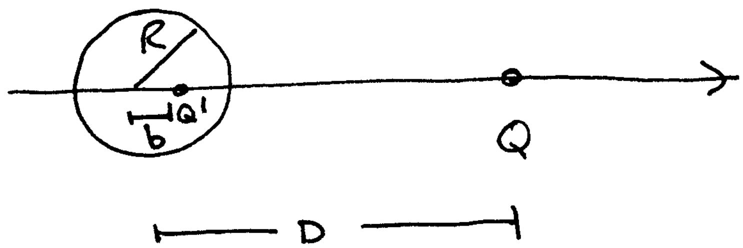
Ex Two charge conducting spheres

$Q = 1\,nC \qquad D = 2\,cm$

$R = \frac{1}{4}\,cm$

Compute force on Q

Approximation I Point charge and charged sphere



Fix boundary condition that surface is all at same potential with image charge.

$$Q' = -\frac{R}{D} Q = -\frac{1}{8} Q$$

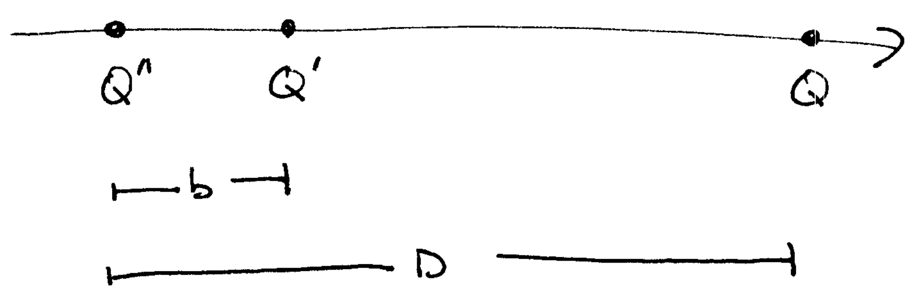
$$b = \frac{R^2}{D} = \frac{\left(\frac{1}{4}\,cm\right)^2}{2\,cm} = \frac{1}{32}\,cm$$

The image solution brings the surface of the sphere to a potential of zero and leaves the sphere with net charge Q'. We want the sphere to have charge Q.

Since the sphere is all at the same potential, any charge added will spread out over the surface evenly. This is like having a point charge at the center. We want total charge Q, so we need a charge $Q'' = Q - Q'$ at the center.

$$Q'' = Q - Q' = Q - \left(-\tfrac{1}{8}Q\right) = \tfrac{9}{8}Q$$

The system we end up with is



The force on Q is

$$F = \frac{kQQ'}{(D-b)^2} + \frac{kQQ''}{D^2}$$

The ratio of this force to the force approximating both as point charges is the error we make with the approximation

$$\frac{F}{F_{point}} = \frac{D^2}{kQ^2}\left(\frac{kQQ'}{(D-b)^2} + \frac{kQQ''}{D^2}\right)$$

$$= \left(\frac{Q'}{Q}\left(\frac{D^2}{(D-b)^2}\right) + \frac{Q''}{Q}\right)$$

$$Q'' = \frac{9}{8}Q \qquad Q' = -\frac{1}{8}Q$$

$$\frac{F}{F_{point}} = \left(-\frac{1}{8}\left(\frac{D^2}{(D-b)^2}\right) + \frac{9}{8}\right)$$

$$= \left(-\frac{1}{8}\left(\frac{(2cm)^2}{(2cm - \frac{1}{32}cm)^2}\right) + \frac{9}{8}\right)$$

$$= \frac{31625}{31752} = 0.996$$

Pith ball electroscope error.

<u>Approximation II</u> Now get the field right on the second sphere. We need images for Q' and Q''. Let them be Q'_2 and Q''_2.

We also need a third charge to bring the total charge of the second sphere to ~~zero~~ Q.

$$Q'''_2 = Q - Q'_2 - Q''_2$$

Now we can compute the next approximation of the force by summing the force of all the charges in sphere 1 on all the charges in sphere 2. Note, the total charge on each sphere is still Q.