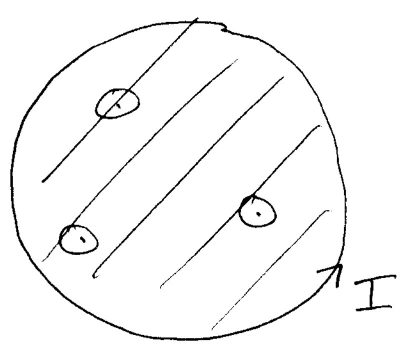
Inductance

The magnetic field is proportional to the current and therefore the flux generated through some path C is proportional to current. Therefore, one might expect the ratio

$$\frac{\Phi_m}{I}$$

to be independent of current.

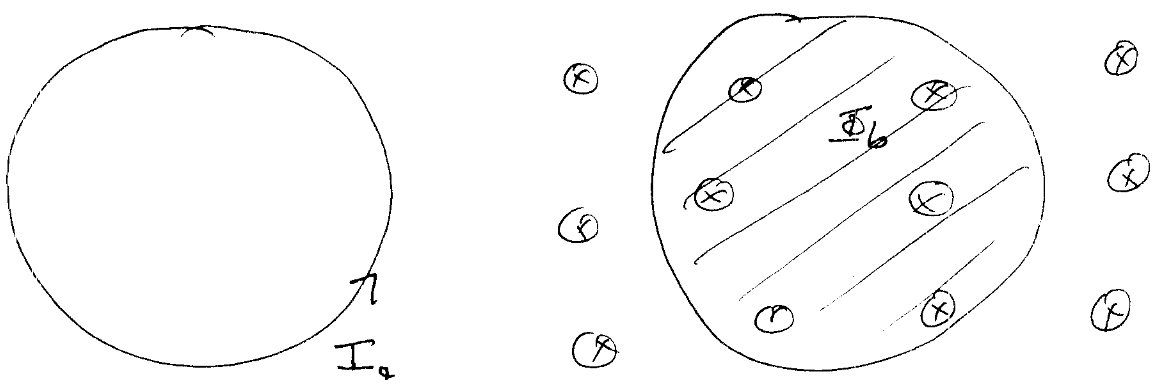
Self-Inductance – (L) - The ratio of the flux through a circuit to the current flowing in a circuit.



$$\frac{\Phi_m}{I} = L$$

Mutual Inductance (M) - The ratio of the flux through one circuit and the field through another circuit

$$M_{ab} = \frac{\Phi_b}{I_a} = M_{ba}$$



Neither M nor L depend on I only on geometry and constants.

Weirdly the ~~inductance~~ mutual inductance is symmetric.

The flux through circuit b is

$$\Phi_b = \oint \vec{A}_a \cdot d\vec{l}_b$$

The vector potential for surface a is

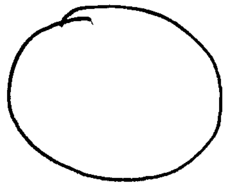
$$\vec{A}_a = \frac{\mu_0 I_a}{4\pi} \oint \frac{d\vec{l}_a}{r}$$

$$\Phi_b = \frac{\mu_0 I_a}{4\pi} \oint_b \oint_a \frac{d\vec{l}_a \cdot d\vec{l}_b}{r}$$

$$M_{ba} = \frac{\Phi_b}{I_a} = \frac{\mu_0}{4\pi} \oint_b \oint_a \frac{d\vec{l}_a \cdot d\vec{l}_b}{r} = M_{ab}$$

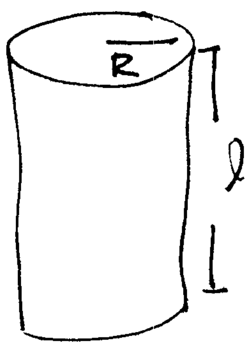
Self- Inductance

Ring



(too hard, magnetic field complicated)

Infinite solenoid / Length



$$B = \mu_0 N' I$$

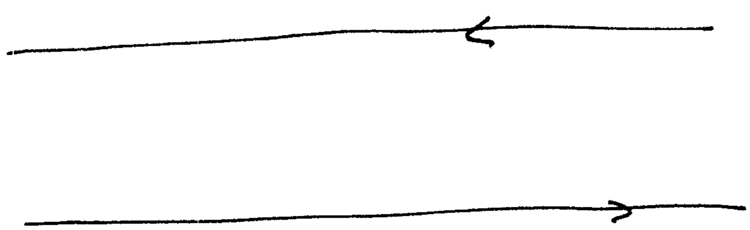
$$\Phi_m = (\pi R^2) N (\mu_0 N' I)$$

$$= NBA$$

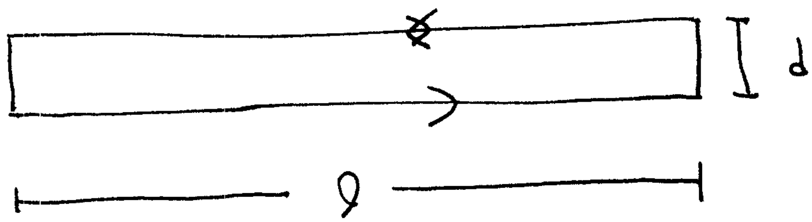
$$L = \frac{\Phi_m}{I} = N A \mu_0 N' \cancel{I}$$

$$= \mu_0 (N')^2 A l$$

Parallel -Wires



Problem : No path – short the wires to produce a closed path.



Let wires have radius r

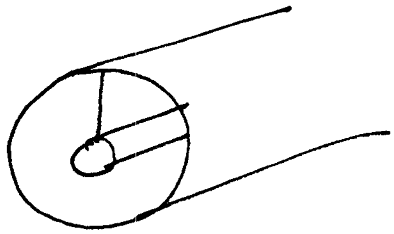
$$B = \frac{\mu_0 I}{2\pi\rho} + \frac{\mu_0 I}{2\pi(d-\rho)}$$

$$\Phi_m = \ell \int B \, d\rho = 2\ell \int_r^{d-r} \frac{\mu_0 I}{2\pi\rho} \, d\rho$$

$$= 2\frac{\mu_0 I \ell}{2\pi} \ln\left(\frac{d-r}{r}\right)$$

$$L = \frac{\Phi_m}{I} = \frac{\mu_0 \ell}{\pi} \ln\left(\frac{d-r}{r}\right)$$

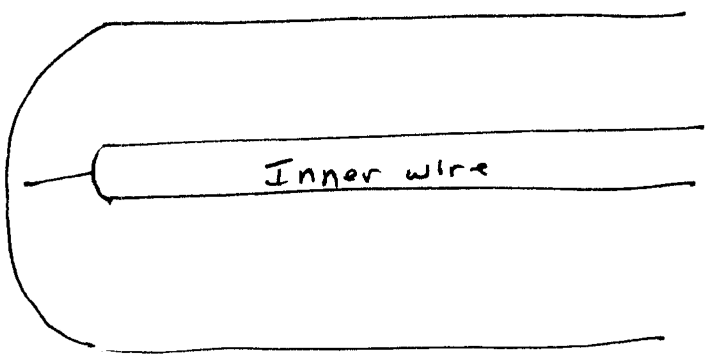
Co-axial Cable



Inner radius r_1

Outer radius r_2

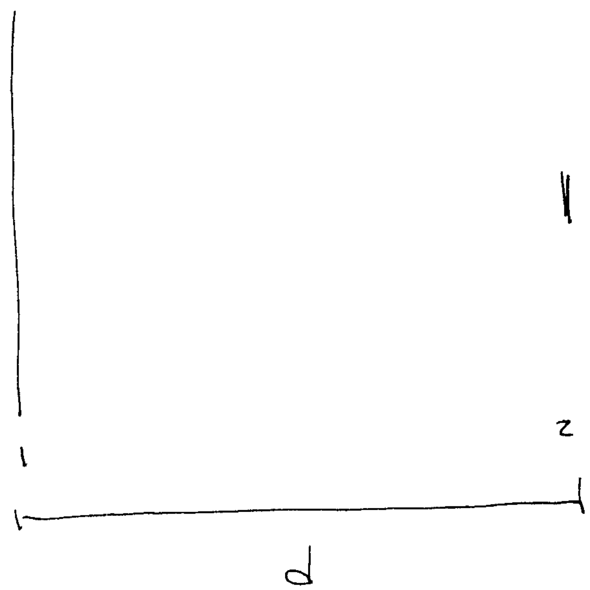
Side view



Connect inner and outer conductors at both ends to form complete circuit.

Mutual Inductance

Ex Compute the mutual inductance of two rings with centers a distance d apart s.t. $R_1 >> R_2$. The rings are co-axial.



Method 1

$$\Phi_2 = B_2 \pi R_2^2 \qquad \text{since } R_2 \text{ small}$$

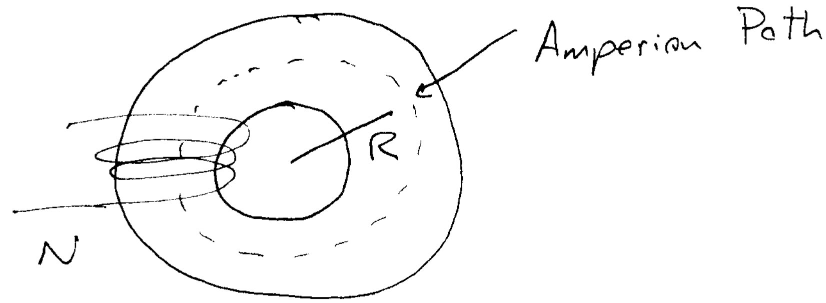
$$B_2 = \frac{\mu_0 I_1 R_1^2}{2(d^2+R_1^2)^{3/2}}$$

$$M_{21} = \frac{\Phi_2}{I_1} = \frac{\mu_0 \pi R_1^2 R_2^2}{2(d^2+R_1^2)^{3/2}}$$

Method 2 Use point dipole formula $m_2 = \pi R_2^2 I_2$ and integrate over ring. Way too hard.

Ex (10.4) ~~Since~~ Iron ring (μ_r), N turns.

Assume field constant. Cross-sectional area πr^2



Compute Field

$$\oint \vec{H} \cdot d\vec{l} = I_f = NI$$

$$2\pi R H$$

$$H = \frac{NI}{2\pi R}$$

$$B = \mu_0 \mu_r H = \frac{\mu_0 \mu_r N I}{2\pi R}$$

Flux $\quad \Phi = NBA = NB\pi r^2$

$$= N\pi r^2 \left(\frac{\mu_0 \mu_r N I}{2\pi R} \right) = \frac{\mu_0 \mu_r N^2 r^2}{2R}$$

Inductance

$$L = \frac{\Phi}{I} = \frac{\mu_0 \mu_r N^2 r^2}{2R}$$

Energy

If a time-varying current runs through an inductor, an emf is produced across the inductor

$$\text{emf} = -\frac{d\Phi}{dt} = -L\frac{dI}{dt} \qquad \text{(Faraday)}$$

The potential difference is defined in terms of the work per unit charge done by an external agent so the voltage that must be applied is

$$V = L\frac{dI}{dt}$$

With this we can calculate the work to set up the field in the inductor

$$\frac{dW}{dt} = \text{Power} = VI$$

$$\text{Energy} = \int dW = \int VI\,dt$$

$$= \int LI\,dI = \frac{1}{2} LI^2$$

$\Rightarrow$ This allows us to compute the energy stored in the field for any situation where we can get the inductance.

Energy Density of Magnetic Field

Calculate energy per unit length stored in a solenoid.

$$U = \frac{1}{2} LI^2 = \frac{1}{2}\left(\mu_0 (N')^2 A \ell\right) I^2$$

$$= \frac{1}{2}(A\ell)\,\frac{(\mu_0 N' I)^2}{\mu_0}$$

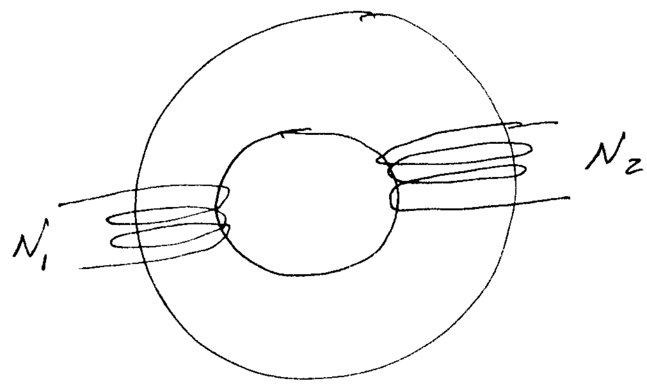
$$= \left(\frac{1}{2}\,\frac{B^2}{\mu_0}\right) \cdot \text{Volume}$$

$$\text{Energy Density} - \frac{1}{2}\,\frac{B^2}{\mu_0} = \frac{1}{2}\vec{B}\cdot\vec{H}$$

This expression is also valid for magnetic materials.

Transformers (Circuit Devices Using Mutual Inductance)

Wind a second loop on the iron ring inductor we used earlier.



The mutual inductance is $\dfrac{\Phi_2}{I_1}$

$$\Phi_2 = N_2 B \pi r^2$$

$$= N_2 \left(\frac{\mu_0 \mu_r N_1 I_1}{2\pi R} \right) \pi r^2$$

$$M_{21} = \frac{\Phi_2}{I_1} = \frac{\mu_0 \mu_r N_1 N_2}{2\pi R} \pi r^2$$

Note, the flux through one turn of the primary (1) is the same as the flux through one loop of the secondary (2).

Let the flux through on loop be Φ_0

$$\Phi_1 = N_1 \Phi_0 \qquad \Phi_2 = N_2 \Phi_0$$

Now let the field change with time,

$$V_1 = \frac{d\Phi_1}{dt} = N_1 \frac{d\Phi_0}{dt}$$

$$V_2 = \frac{d\Phi_2}{dt} = N_2 \frac{d\Phi_0}{dt}$$

$$\boxed{\frac{V_2}{V_1} = \frac{N_2}{N_1}}$$

Transformers step-up or step-down voltage.

Transformers are also very efficient so

$$\text{Power}_1 = \text{Power}_2$$

$$V_1 I_1 = V_2 I_2$$