Electromagnetic Force

The electric and magnetic fields exert a force on a charged particle given by the Lorentz force.

$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$$

where $\vec{v}$ is the velocity of the particle.

If the charged particles are moving such that they can be described as a current

$$d\vec{F} = I\,d\vec{l}\times\vec{B} = d\vec{I}\times\vec{B}$$

Naturally, these expressions can be extended to a force per unit area or a force per unit volume.

The electric force is straight forward. If the charge is positive, the force and field are parallel.

The electric force behaves just like the gravitational force:

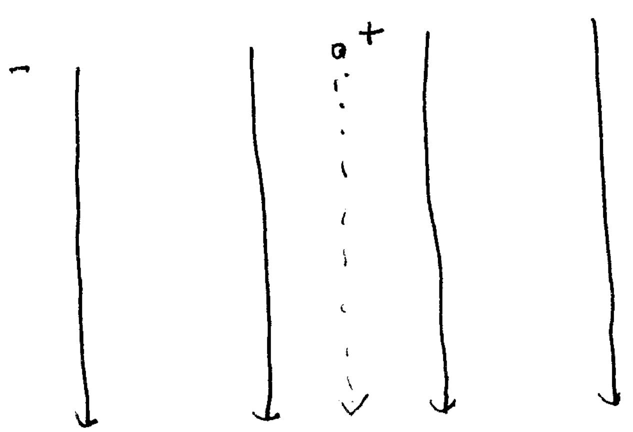
Uniform $\quad \vec{F} = m\vec{g} \iff \vec{F} = q\vec{E}$

or

$$\vec{F} = \frac{mMG}{r^2}\hat{r} \iff \vec{F} = \frac{q_1 q_2 k}{r^2}\hat{r}$$

So if the field is uniform, all the stuff you know about motion near the earth applies.

Let $\vec{E} = -E_0\hat{z}$

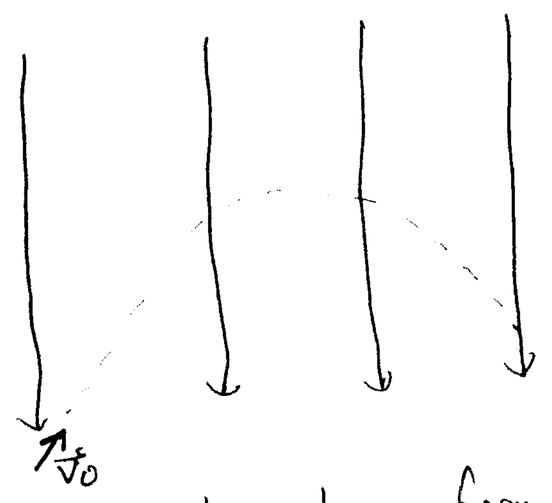


Release Positive Charge, the distance travelled is

$$d = d_0 + V_0 t + \frac{1}{2}at^2$$

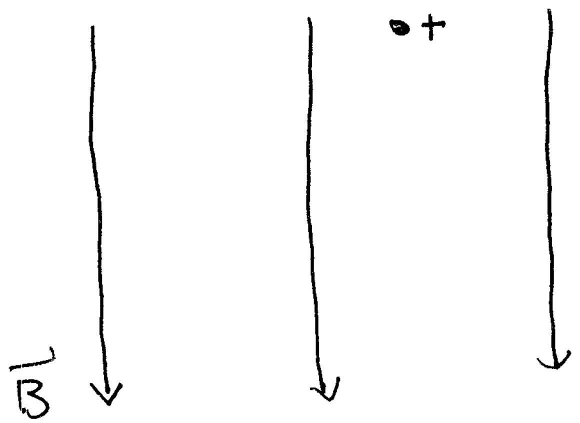
$$F = qE = ma$$

$\vec{E} = -E_0\hat{z}$



Fire particle into uniform field. We get the same parabolic trajectories we found in UPI.

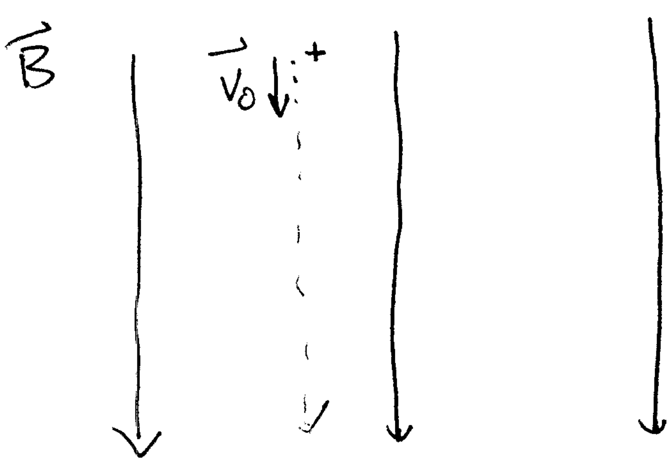
Magnetic fields are different



If we release a particle
it just sits there.

If $v_0 = 0$, $F = 0$

If we fire a particle in the direction of the field, the particle travels un-deflected, and unaccelerated

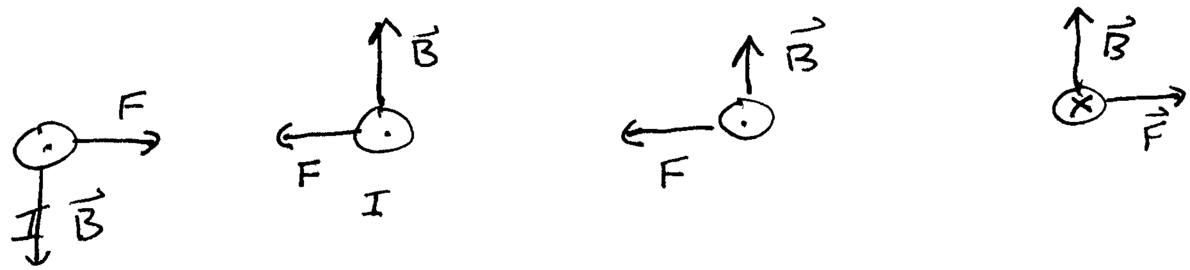


$\vec{F} = 0$, $\vec{V}$ = constant

A bit of a segue

For electric charges, likes repel / opposites attract.
What about currents? Consider two long wires.

$$B = \frac{\mu_0 I}{2\pi\rho} \qquad F = ILB$$
$$\vec{F} = I\vec{L} \times \vec{B}$$



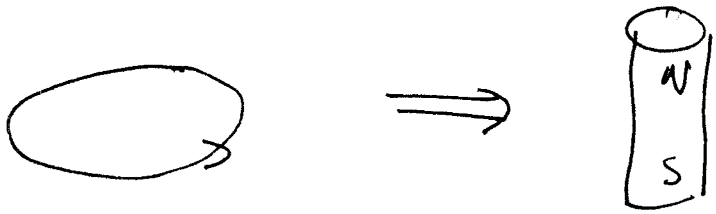
Like currents attract, opposites repel.

Now, wrap the current in a ring

Rings attract

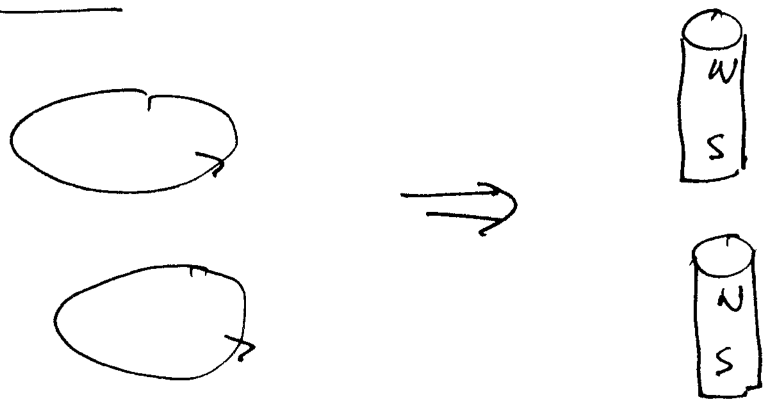
Rings repel.

But all a permanent magnet is, is a bunch of aligned current loops



(check the fields to ensure my direction is correct)

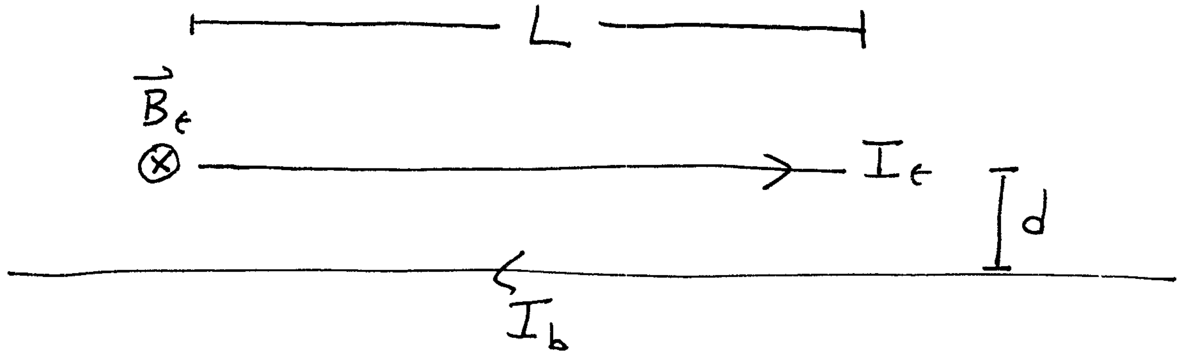
Therefore



Attract Attract

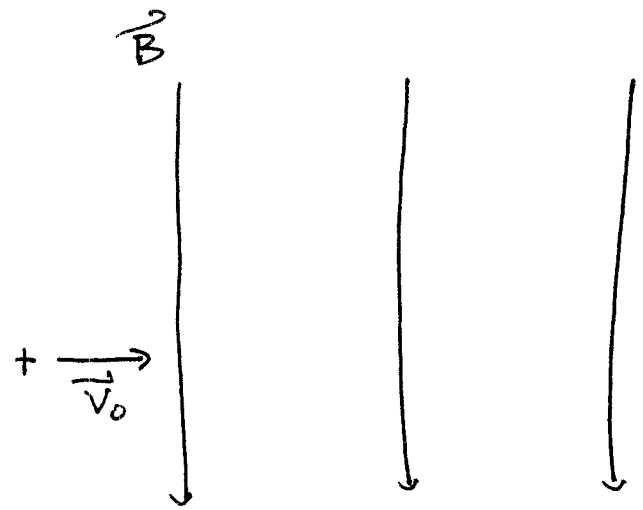
$\Rightarrow$ Unlike poles attract, like poles repel.

A classic experiment, the current balance



$$\vec{B}_t = \frac{\mu_0 I_b}{2\pi d} \text{ into page}$$

$$|\vec{F}| = I_t L B = \frac{\mu_0 I_t I_b L}{2\pi d}$$

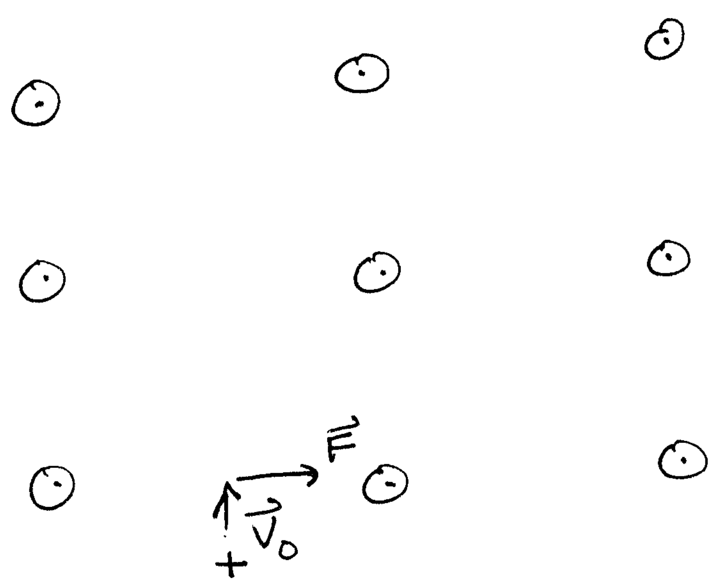


Now, shoot a positive charge $\perp$ field.

$$\vec{F} = q\vec{v} \times \vec{B}$$

$\vec{F}$ into page

Let's look at that from another angle.



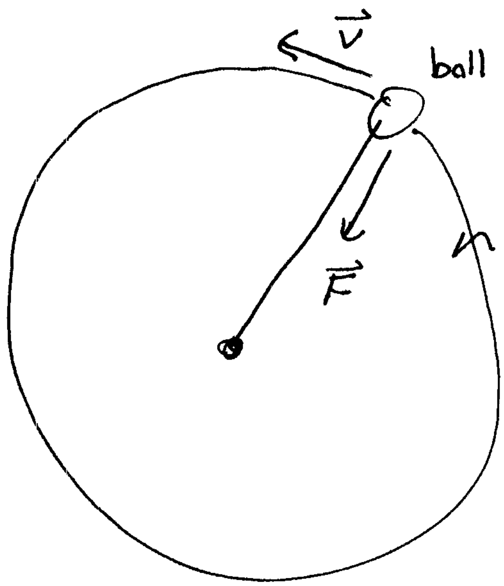
$\vec{B}$ out of page

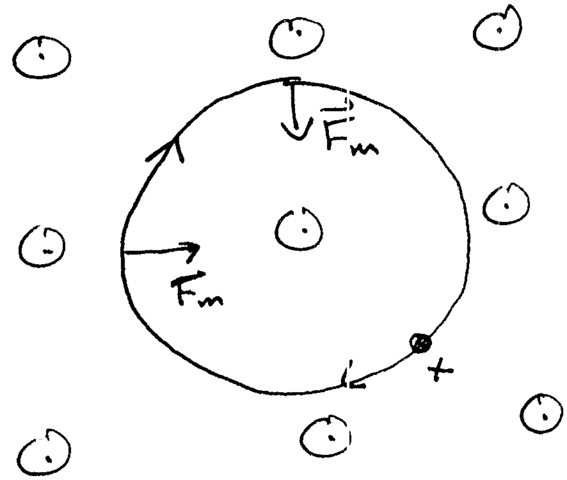
$\vec{F} \perp \vec{v} \implies W_m = 0$ since $\vec{F} \perp d\vec{r}$

and $W_m = \int \vec{F} \cdot d\vec{r}$

- Magnetic field does not do work.
- Magnetic field cannot change the energy of a particle.
- Magnetic field can change the direction but not the speed of a particle.

So we have a particle moving at constant velocity experiencing a force perpendicular to its velocity. This is exactly the situation we have when we spin a ball on a rope





The acceleration (centripetal) of anything moving in a circle is

$$a_c = \frac{v^2}{r} = r\omega^2$$

where ω is the angular velocity $\omega = \frac{d\phi}{dt}$

The magnetic force on the particle is

$$|\vec{F}| = |q\vec{v} \times \vec{B}| = qvB$$

Newton's II Law

$$F = ma$$

$$qvB = m\frac{v^2}{r} = mr\omega^2$$

$$r = \frac{mv}{qB} \quad \text{or} \quad \omega = \frac{qB}{m}$$

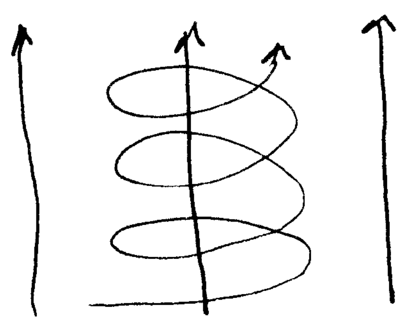
As usual, the linear frequency of the orbit is

$$f = \frac{\omega}{2\pi}$$

and the period of the orbit ~~is~~ is $T = \frac{1}{f}$.

The frequency ω is called the <u>cyclotron</u> frequency.

What happens if we have an electric and magnetic field in the same region and the fields are parallel

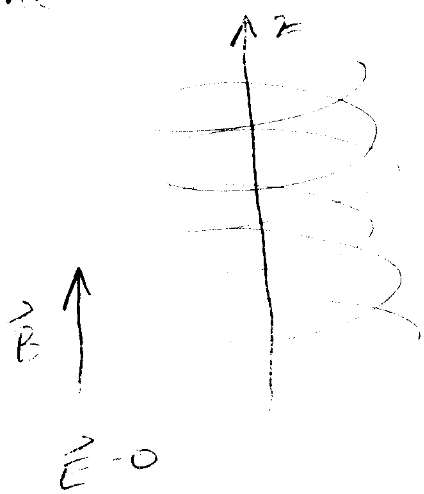


The particle travels in a helix.

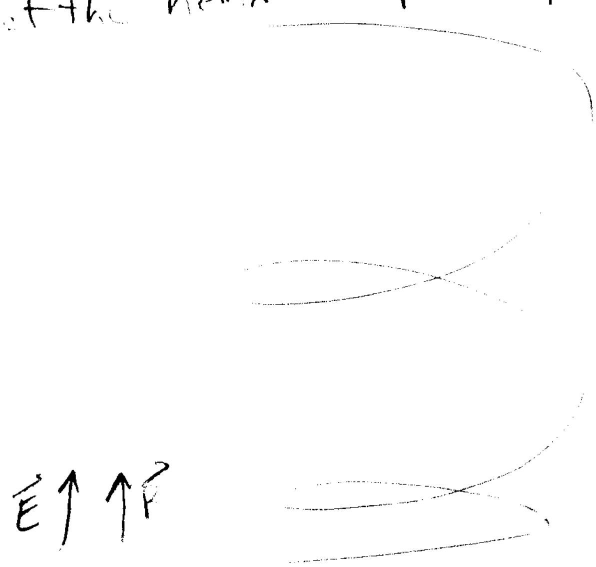
Both $\vec{E}$ and $\vec{B}$ together

Let's be a little more careful with the helix.

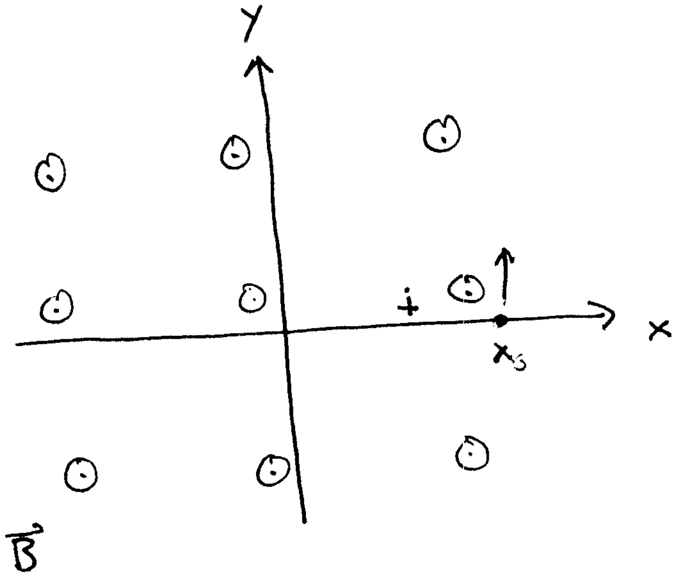
$\vec{B}$ uniform, $\vec{E} = 0$, but $v_{z0} \neq 0$. So the particle has an initial velocity with a component parallel to the field. This also yields a helix with constant v_z



If $\vec{B}$ uniform, $\vec{E}$ uniform, and $\vec{B} \parallel \vec{E}$ the spacing of the helix wraps expands.



Let's do the circular motion correctly.



Initial conditions

$\vec{v}_0 = (0, v_0, 0)$

$\vec{r}_0 = (R, 0, 0)$

$\vec{B} = B_0 \hat{z}$

Newton's II Law

$$\vec{F} = q\vec{v} \times \vec{B} = m\vec{a}$$

$$\vec{F} = q \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ 0 & 0 & B_0 \end{vmatrix} = q \hat{x} v_y B_0 - q \hat{y} v_x B_0$$

<u>EOM</u>

<u>x-component</u>

$$m a_x = q v_y B_0$$

<u>y-component</u>

$$m a_y = -q v_x B_0$$

8

$$m a_x = m\frac{dv_x}{dt} = q v_y B_0 \quad \Big| \quad m a_y = m\frac{dv_y}{dt} = -q v_x B_0$$

Simplify using cyclotron frequency $\omega = \frac{qB_0}{m}$

$$\frac{dv_x}{dt} = \omega v_y \quad \Big| \quad \frac{dv_y}{dt} = -\omega v_x$$

Differentiate and substitute

$$\frac{d^2 v_x}{dt^2} = \omega \frac{dv_y}{dt} = -\omega^2 v_x \quad \Big| \quad \frac{d^2 v_y}{dt^2} = -\omega \frac{dv_x}{dt} = -\omega^2 v_y$$

Recognize the simple harmonic oscillator (SHO)

$$v_x = A\cos\omega t + B\sin\omega t \quad \Big| \quad v_y = C\cos\omega t + D\sin\omega t$$

$$\frac{dv_x}{dt} = -\omega A\sin\omega t + \omega B\cos\omega t = \omega v_y = \omega C\cos\omega t + \omega D\sin\omega t$$

$$\Rightarrow D = -A \quad , \quad B = C$$

So $v_x = A\cos\omega t + B\sin\omega t \quad \Big| \quad v_y = B\cos\omega t - A\sin\omega t$

Initial condition, $\vec{v}_0 = (0, v_0, 0) \Rightarrow A = 0 \, , \, B = v_0$

$$v_x = v_0 \sin\omega t \qquad v_y = -v_0 \cos\omega t$$

Integrate to get position

$$x(t) = -\frac{v_0}{\omega}\cos\omega t + C_1$$

$$y(t) = -\frac{v_0}{\omega}\sin\omega t + C_2$$

Let $v_0 = \omega R$

<u>Initial conditions</u> $\vec{r_0} = (x_0, 0, 0)$

$$x(0) = -R + C_1 = x_0$$

$$y(0) = C_2 = 0$$

$$C_2 = 0 \qquad C_1 = x_0 + R$$

<u>Solution</u>

$$x(t) = -R\cos\omega t + x_0 + R$$

$$y(t) = -R\sin\omega t$$

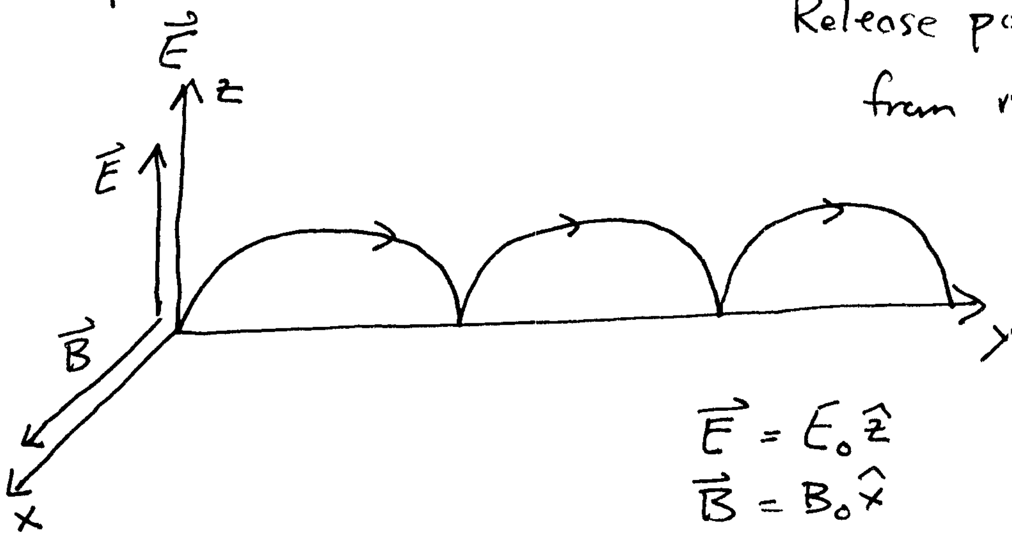
<u>Trajectory</u>

$$(x - (x_0 + R))^2 + y^2 = R^2(\cos^2\omega t + \sin^2\omega t) = R^2$$

$\Longrightarrow$ Circle of radius R with center $x_0 + R$

Now let's try a crossed electric and magnetic field. This is done quantitatively in Griffith's 5.2 and qualitatively in one of the physics subject GRE practice tests.

Release particle from rest.



$$\vec{E} = E_0 \hat{z}$$
$$\vec{B} = B_0 \hat{x}$$

The force in the x-direction is zero, so the trajectory is described by

$$\vec{r}(t) = (0,\ y(t),\ z(t))$$

The force is $\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$

$$q\vec{v}\times\vec{B} = q \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & v_y & v_z \\ B_0 & 0 & 0 \end{vmatrix} = v_z B_0 \hat{y} q - v_y B_0 \hat{z} q$$

$$\vec{F} = qE_0\hat{z} - qv_yB_0\hat{z} + qv_zB_0\hat{y}$$

$$= m\vec{a}$$

$$\vec{a} = \left(\frac{qE_0}{m} - v_y\omega\right)\hat{z} + \omega v_z\hat{y} \qquad \omega = \frac{qB}{m}$$

$$= \omega\left(\frac{E_0}{B_0} - v_y\right)\hat{z} + \omega v_z\hat{y}$$

EOM

$$\frac{dv_y}{dt} = a_y = \omega v_z$$

$$\frac{dv_z}{dt} = -\omega v_y + \omega\frac{E_0}{B_0}$$

$$\frac{d^2v_z}{dt^2} = -\omega\frac{dv_y}{dt} = -\omega^2 v_z$$

$$\frac{d^2v_y}{dt^2} = \omega\frac{dv_z}{dt} = \cancel{-\omega^2\frac{dv_y}{dt} + \omega^2\frac{E_0}{B_0}}$$

$$= -\omega^2 v_y + \omega^2\frac{E_0}{B_0}$$

Solution

$$v_y = A \sin\omega t + B \cos\omega t + E_0/B_0$$

$$v_z = C \sin\omega t + D \cos\omega t$$

but $$\frac{dv_y}{dt} = A\omega \cos\omega t - B\omega \sin\omega t = \omega v_z$$

$$D = A \quad \text{and} \quad C = -B$$

$$v_y = A \sin\omega t + B \cos\omega t + E_0/B_0$$

$$v_z = -B \sin\omega t + A \cos\omega t$$

Integrate

$$y(t) = -\frac{A}{\omega} \cos\omega t + \frac{B}{\omega} \sin\omega t + \frac{E_0}{B_0} t + C_1$$

$$z(t) = \frac{B}{\omega} \cos\omega t + \frac{A}{\omega} \sin\omega t + C_2$$

Initial Conditions

$$V_y(0) = 0 \quad \Rightarrow \quad B + \frac{E_0}{B_0} = 0 \qquad B = -\frac{E_0}{B_0}$$

$$V_z(0) = 0 \quad \Longrightarrow \quad A = 0$$

$$y(t) = -\frac{E_0}{B_0 \omega} \sin\omega t + \frac{E_0}{B_0} t + C_1$$

$$z(t) = -\frac{E_0}{B_0 \omega} \cos\omega t + C_2$$

$$y(0) = 0 \quad \Longrightarrow \quad C_1 = 0$$

$$z(0) = 0 \quad \Longrightarrow \quad C_2 = \frac{E_0}{B_0 \omega}$$

$$y(t) = \frac{E_0}{B_0 \omega}\left(\omega t - \sin\omega t\right)$$

$$z(t) = \frac{E_0}{B_0 \omega}\left(1 - \cos\omega t\right)$$

Eliminate sin and cos and let $R = \frac{E_0}{B_0 \omega}$

$$(y - R\omega t)^2 + (z - R)^2 = R^2$$

Circle with center $(0, R\omega t, R)$

speed of center in y-direction $v = R\omega = \frac{E_0}{B_0}$