Integral Theorems

UP II $\rightarrow$ PHYS 3413 by divergence + Stokes's Thm

Gradient $\quad df = (\nabla f)\cdot d\vec{l}$

$$f(\vec{b}) - f(\vec{a}) = \int_{\vec{a}}^{\vec{b}} (\nabla f)\cdot d\vec{l}$$

$$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$(\nabla f)\cdot d\vec{l} = \left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z}\right)dz = df$$

* Note, depends only on endpoints

Flux $\quad$ The flux out of a closed surface S is

$$\Phi = \oint \vec{A}\cdot d\vec{a}$$

where $d\vec{a}$ is an element of area da multiplied by the outward surface normal $\hat{n}$.

Divergence (Green's) Thm The total divergence of a field in a volume, V, is equal to the flux of the field out of the surface, S, bounding the volume.

$$\int_V (\nabla \cdot \vec{A})\, dv = \oint_S \vec{A} \cdot d\vec{a}$$

Stoke's Thm The integral of a field $\vec{A}$ around a closed curve C is equal to the total curl of the field over the surface S bounded by C

$$\oint_C \vec{A} \cdot d\vec{l} = \int_S (\nabla \times \vec{A}) \cdot d\vec{a}$$

$\Rightarrow$ The normal to S is chosen so ~~the curve~~ ~~C traces~~ if the fingers of the right hand point in the direction C is traced the thumb points in the direction of $\hat{n}$.

Other integral thms

$$\int_V (\nabla f)\, dv = \oint_S f\, d\vec{a}$$

$$\int_S \nabla f \times d\vec{a} = -\oint_C f\, d\vec{l}$$

$$\int_V (\nabla \times \vec{A})\, dv = -\oint_S \vec{A} \times d\vec{a}$$

$$\int_S (d\vec{a} \times \nabla) \times \vec{A} = -\oint \vec{A} \times d\vec{\lambda} \qquad \text{(Stoke's 2)}$$

$$\int_V (f \nabla^2 g + \nabla f \cdot \nabla g)\, dv = \int_S (f \nabla g) \cdot d\vec{a} \qquad \text{(Green 1)}$$

$$\int_V (f \nabla^2 g - g \nabla^2 f)\, dv = \int_S (f \nabla g - g \nabla f) \cdot d\vec{a}$$

(Green 2)

What does this stuff mean?

The math we use in this class is used in virtually all areas of physics and engineering. The systems change, but the math stays.

Gradient – ∇f points in the direction f is changing most rapidly.

Consider – Heat Conduction (Thermal Energy Conduction)

~~Heat~~ Thermal energy flows from high temperature to low temperature, so it flows in the direction $-\nabla T$. The amount of thermal energy that flows depends on the size of the temperature gradient and the thermal conductivity K. The flux of thermal energy, the thermal current is then $\vec{J} = -K\nabla T$

The internal energy of a system is

$$U = mcT$$

where m is the mass and c the specific heat.

The internal energy density $u = \rho c T$ where ρ is the mass density.

Conservation of energy requires that the time rate of change of the energy in a volume must be equal and opposite the energy flowing out of the volume.

$$\int_V \frac{\partial u}{\partial t} dv = \frac{\partial}{\partial t}\int_V u\, dv = -\int_S \vec{J}\cdot d\vec{a}$$

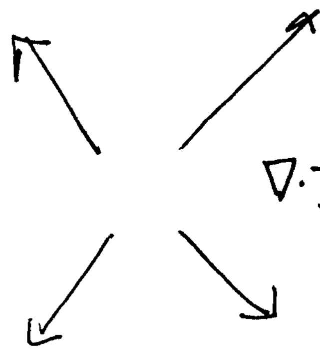
$$= -\int_V (\nabla\cdot\vec{J})\, dv \qquad \text{(divergence Thm)}$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} + \nabla\cdot\vec{J} = 0 \qquad \underline{\text{continuity eqn}}$$

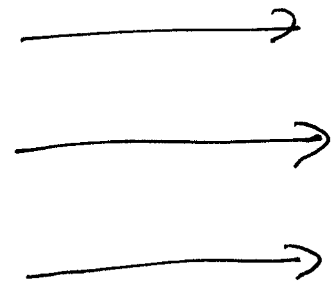
$$\rho c \frac{\partial T}{\partial t} - K\nabla\cdot\nabla T = 0$$

$$\nabla^2 T = \frac{\rho c}{K}\frac{\partial T}{\partial t}$$

The divergence represents the amount of net "flow" of the field out of the volume. The net number of field lines leaving a closed surface.



$\nabla \cdot \vec{J} \neq 0$



$\nabla \cdot \vec{J} = 0$

"solenoidal"

Now consider curl and fluid flow

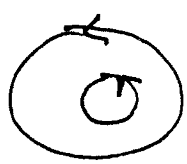
The flux of fluid $\vec{J} = \rho \vec{v}$

density

velocity

<u>Continuity</u> $$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$$

but that's not the whole story. Fluids develop whirlpools

 is "solenoidal"

A whirlpool exists if

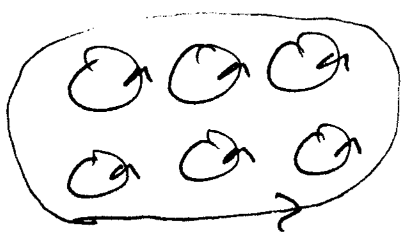
$$\oint_C \vec{A} \cdot d\vec{l} \neq 0$$

$$= \int_S (\nabla \times \vec{A}) \cdot d\vec{a} \qquad \text{Stoke's}$$

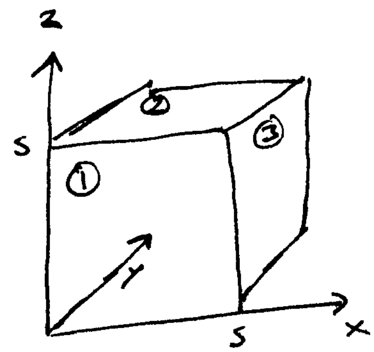
$\Rightarrow$ so $\nabla \times \vec{A}$ measures the local rotation of the field

$\Rightarrow$ if $\nabla \times \vec{A} = 0$ the field is "~~ir~~ irrotational".

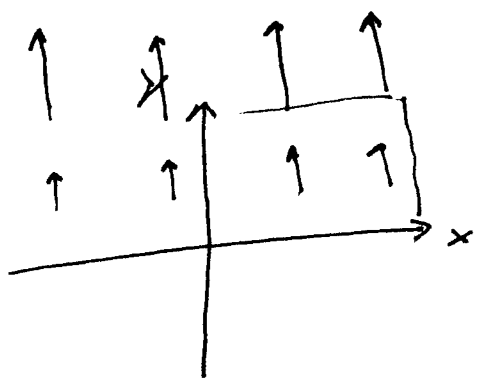
A combination of little whirls make a big whirl.



T-23 Demonstrate Divergence Thm for vector $\vec{A} = y\hat{y}$ for a cube of side s with corner at origin and sides of length s.



Primed sides opposite numbered sides.



$n_1 = -\hat{y}$

$n_{1'} = \hat{y}$

$n_2 = \hat{z}$

$n_{2'} = -\hat{z}$

$\hat{n}_3 = \hat{x}$

$\hat{n}_3' = -\hat{x}$

$$\oint_S \vec{A}\cdot d\vec{a} = \int_1 + \int_{1'} + \int_2 + \int_{2'} + \int_3 + \int_{3'}$$

$$= \underset{A=0}{0} + \int \vec{A}\cdot\hat{y}\,da + \underbrace{0 + 0 + 0 + 0}_{\vec{A}\perp\hat{n}}$$

$$= s^2(A(s)) = s^3$$

$$\int_V (\nabla \cdot \vec{A})\, dv$$

$\nabla \cdot \vec{A} = 1$ $\qquad$ $\int_V (\nabla \cdot \vec{A})\, dv = V = s^3$

so $\int_V (\nabla \cdot \vec{A})\, dv = \int_S \vec{A} \cdot d\vec{a}$ ✓

1.25 $$\oint d\vec{a} = \oint_S d\vec{a}$$

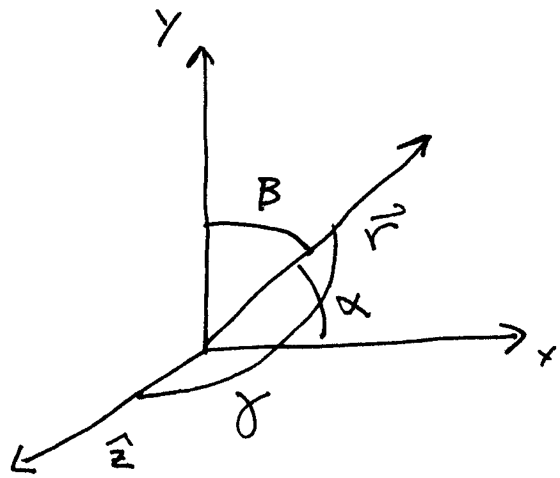
where S is a closed surface

$$\int_V \nabla f \, dv = \oint_S f \, d\vec{a}$$

Let $f = 1$, $\nabla f = 0$

$$\oint_S d\vec{a} = 0$$

(1-30) Direction Cosines



$\hat{x} \cdot \vec{r} = r\cos\alpha \qquad \hat{y} \cdot \vec{r} = r\cos\beta$

$\hat{z} \cdot \vec{r} = r\cos\gamma$

$\hat{r} \cdot \hat{x} = \cos\alpha \qquad \hat{r} \cdot \hat{y} = \cos\beta \qquad \hat{r} \cdot \hat{z} = \cos\gamma$

$\hat{r} = (\cos\alpha, \cos\beta, \cos\gamma)$

$\hat{r} \cdot \hat{r} = 1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$