Return to div, grad, curl

A couple Thm's about $\nabla\cdot$ and $\nabla\times$

Thm 1 Irrotational Fields ($\nabla\times\vec{A}=0$). The following are equivalent

A) $\nabla\times\vec{A}=0$ everywhere

B) $\int_{\vec{a}}^{\vec{b}}\vec{A}\cdot d\vec{l}$ is independent of path

C) $\oint\vec{A}\cdot d\vec{l}=0$ for any closed loop.

D) $\vec{A}=\nabla f$ for some scalar function f

Thm 2 Solenoidal Fields - $\nabla\cdot\vec{A}=0$ everywhere.

The following are equivalent

A) $\nabla\cdot\vec{A}=0$ everywhere

B) $\int\vec{A}\cdot d\vec{a}$ is independent of surface for any curve C.

C) $\oint\vec{A}\cdot d\vec{a}=0$ for any closed curve

D) $\vec{A}=\nabla\times\vec{B}$ for some vector field $\vec{B}$

For any $\vec{A}$, $\quad \vec{A} = -\nabla f + \nabla \times \vec{B}$

for some scalar function f and some vector function $\vec{B}$.

<u>Coulomb's Law for Electric Force</u> – The force q at $\vec{r}' = (x', y', z')$ exerts on Q at $\vec{r} = (x, y, z)$ is

$$\vec{F}(\vec{r}, \vec{r}') = \frac{qQ}{4\pi\varepsilon_0 (r'')^2} \hat{r}''$$

<u>Displacement (Separation) Vector</u> – Vector from q to Q

$$\vec{r}'' = \vec{r} - \vec{r}' = (x-x', y-y', z-z') = \vec{\mathscr{r}}$$

(Griffiths)

$$r'' = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

__Dfn Electric Field ($\vec{E}$)__ - The electric field $\vec{E}$ is the electric force per unit charge a small test charge would feel.

$$\vec{E} = \frac{\vec{F}}{Q}$$

__Coulomb's Law__ - Electric Field of Point Charge - The electric field at $\vec{r}$ due to a point charge q at $\vec{r}'$ is

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 (r'')^2} \hat{r}''$$

where $\varepsilon_0 \equiv$ Permitivity of Free Space

$$= 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2}$$

$$= 1 / \left[(299792458)^2 \cdot 4\pi \times 10^{-7} \right] \frac{C^2}{N \cdot m^2}$$

exactly.

Let's work on the field of a point charge at the origin. For this charge $\vec{r}' = 0$ and $\vec{r}'' = \vec{r}$. The results we derive will be general since the physics can't be affected by a shift of origin.

$$\nabla \cdot \vec{E} = \nabla \cdot \left[\frac{q}{4\pi\varepsilon_0 r^2} \hat{r} + 0\hat{\phi} + 0\hat{\theta} \right]$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{q}{4\pi\varepsilon_0 r^2} \right) = 0$$

Electric field in a charge free region is solenoidal.

<u>Divergence Thm</u>

$$\int_V (\nabla \cdot \vec{E})\, dv = \oint_S \vec{E} \cdot d\vec{a}$$

$$= \frac{q}{4\pi\varepsilon_0} \oint_S \frac{\hat{r}}{r^2} \cdot \hat{r}\, da$$

$$= \frac{q}{4\pi\varepsilon_0 r^2} \oint da = \frac{q(4\pi r^2)}{4\pi\varepsilon_0 r^2} = \frac{q}{\varepsilon_0}$$

This result holds in general and does not depend on the locaton of the charge in the volume.

<u>Gauss' Law</u> The flux exitting a closed surface S is proportional to the charge Q enclosed in the surface

$$\Phi \equiv \oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

Let $\rho(\vec{r})$ be the charge density in volume V

$$Q_{enc} = \int_V \rho(\vec{r})\, dv$$

$$\int_V \nabla \cdot \vec{E}\, dv = \oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} \int_V \rho\, dv$$

for any volume

<u>Gauss' Law</u> – Differential form

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

We have a bit of a mystery.

$\nabla \cdot \vec{E} = f$ is a function such that

(A) $f = 0$

(B) $\int f \, dv = \frac{q}{\varepsilon_0}$

Something odd must be happening at $r=0$.

$$\nabla \cdot \vec{E} = \frac{q}{4\pi\varepsilon_0} \nabla \cdot \left(\frac{\hat{r}}{r^2} \right)$$

$$\frac{q}{4\pi\varepsilon_0} \int_V \nabla \cdot \left(\frac{\hat{r}}{r^2} \right) dv = \frac{q}{\varepsilon_0}$$

$$\boxed{\int_V \nabla \cdot \left(\frac{\hat{r}}{r^2} \right) dv = 4\pi}$$

<u>Dirac Delta Function</u> (Distribution) $\delta(\vec{r})$

Function that is zero everywhere except at $\vec{r} = 0$, s.t.

$$\int \delta(\vec{r}) \, dv = \cancel{4}$$
$$= 1$$

Therefore

$$\nabla \cdot \left(\frac{\hat{r}}{r^2} \right) = 4\pi \delta(\vec{r})$$

This is a three-dimensional delta function,

$$\delta(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z)$$

where

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

Think of the δ function as the following limit

$$\delta_a(x)$$

The area is 1. If we take the limit $a \to 0$, the area is still 1.

Now consider

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx$$

If a is sufficiently small, $f(x)$ changes slowing over the range $\delta_a(x)$ is non-zero, so it can be removed from the integral.

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0) \int_{-\infty}^{\infty} \delta(x)\, dx$$

$$= f(0)$$

or more generally

$$\int_{-\infty}^{\infty} f(x) \delta(x-b)\, dx = f(b)$$

- The integral would be zero if the range did not include b.

- To prove something about a delta function, we show it is true for an arbitrary function f.

Ex Evaluate

$$\int_0^2 x^2 e^{-x}\,\delta(x-1)\,dx = (1)^2 e^{-1}$$

$$\int_3^4 x^2 e^{-x}\,\delta(x-1)\,dx = 0$$

Ex What is $\delta'(x)$?

$$\int f\delta'(x)\,dx = f\delta - \int \delta(x) f'(x)\,dx$$

$$= -f'(0)$$

using integration by parts

Delta functions have a great variety of additional properties and representations

for example

$$\delta(ax) = \frac{\delta(x)}{|a|} \quad \text{or} \quad \delta(x) = \int_{-\infty}^{\infty} e^{-2\pi i k x}\,dk$$

Let's generalize a bit, if

$$\nabla \cdot \left(\frac{\hat{r}}{r^2} \right) = 4\pi \delta(\vec{r})$$

then

$$\nabla \cdot \left(\frac{\hat{r}''}{(r'')^2} \right) = 4\pi \delta(\vec{r}'')$$

$$= 4\pi \delta(\vec{r} - \vec{r}')$$

where $\vec{r}'' = \vec{r} - \vec{r}'$

The field of a point charge is irrotational

$$\oint \vec{E} \cdot d\vec{l} = \oint \frac{q}{4\pi\varepsilon_0 r^2} \hat{r} \cdot \left(dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi} \right)$$

$$= \int_{\vec{a}}^{\vec{b}} \frac{q\,dr}{4\pi r^2 \varepsilon_0}$$

$$= \frac{q}{4\pi\varepsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right)$$

If a closed path is used, $\oint \vec{E} \cdot d\vec{l} = 0$

and $\nabla \times \vec{E} = 0$ (from Stoke's Thm)

By our general properties of the curl, there exists a scalar function, $V(\vec{r})$, such the $\vec{E} = -\nabla V(\vec{r})$. This function will be called the <u>electric potential.</u>

From the simple form of $\vec{E}$ for a point charge, we can guess $V(r) = \dfrac{q}{4\pi\varepsilon_0 r} + C$. It is convienient to set $V(\infty) = 0 \Rightarrow C = 0$. With this choice, the potential of a point charge is

$$V(r) = \frac{q}{4\pi\varepsilon_0 r} \quad \text{or} \quad V(r) = \frac{q}{4\pi\varepsilon_0 r''}$$

By observation, $\dfrac{\hat{r}}{r^2} = -\nabla\left(\dfrac{1}{r}\right)$ so

$$\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = -\nabla^2\left(\frac{1}{r}\right) = 4\pi\sigma(\vec{r})$$

or

$$\nabla\cdot\left(\frac{\hat{r}''}{(r'')^2}\right) = -\nabla^2\left(\frac{1}{r''}\right) = 4\pi\sigma(r'')$$

What does this get us?

$$\vec{E} = -\nabla V$$

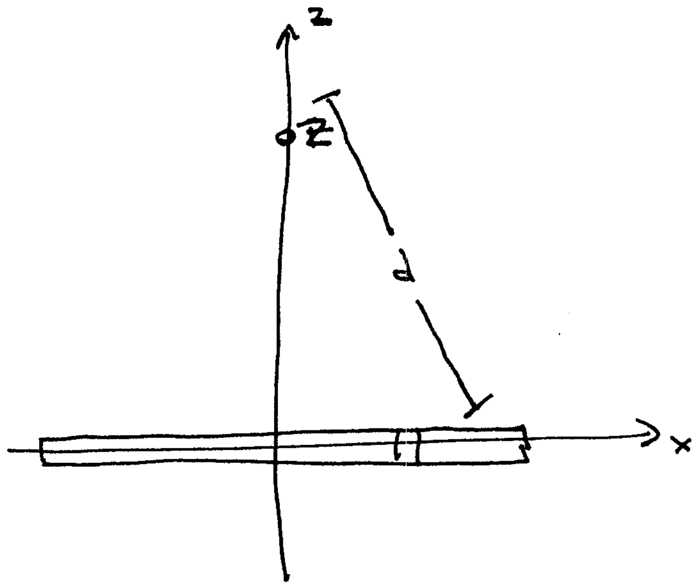
$$\nabla \cdot \vec{E} = -\nabla^2 V = \rho/\varepsilon_0$$

$$\nabla^2 V + \frac{\rho}{\varepsilon_0} = 0 \qquad \text{Poisson's Eqn}$$

Differential equation we will be solving continuously in the second part of the class.

Before we start solving differential equations, note potential exceptionally simplifies the calculation we have been doing.

Ex Compute the electric field of a finite line of charge from $x=-L$ to $x=L$ with charge density λ.



$d = \sqrt{x^2 + z^2}$ $\quad\quad dq = \lambda d\cancel{x}$

$$V(z) = \int_{-L}^{L} \frac{k\lambda\, dx}{\sqrt{x^2+z^2}} =$$

$$k\lambda\left[-\ln\left(-L+\sqrt{L^2+z^2}\right) + \ln\left(L+\sqrt{L^2+z^2}\right)\right]$$

$$\vec{E} = -\nabla V = -\frac{dV}{dz}\hat{z}$$

$$= \frac{2\lambda L k}{z\sqrt{L^2+z^2}}\hat{z}$$

$$\rightarrow \vec{E} = \frac{2\lambda k}{z}\hat{z} \quad \text{as} \quad L \to \infty$$

> $integrate\left(\frac{1}{sqrt\left(x^2+z^2\right)},\ x=-L..L\right);$

$$-\ln\left(-L+\sqrt{L^2+z^2}\right)+\ln\left(L+\sqrt{L^2+z^2}\right) \tag{1}$$

> $simplify(\%);$

$$-\ln\left(-L+\sqrt{L^2+z^2}\right)+\ln\left(L+\sqrt{L^2+z^2}\right) \tag{2}$$

> $diff(\%,\ z);$

$$-\frac{z}{\sqrt{L^2+z^2}\left(-L+\sqrt{L^2+z^2}\right)}+\frac{z}{\sqrt{L^2+z^2}\left(L+\sqrt{L^2+z^2}\right)} \tag{3}$$

> $simplify(\%);$

$$-\frac{2\,L}{z\sqrt{L^2+z^2}} \tag{4}$$

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