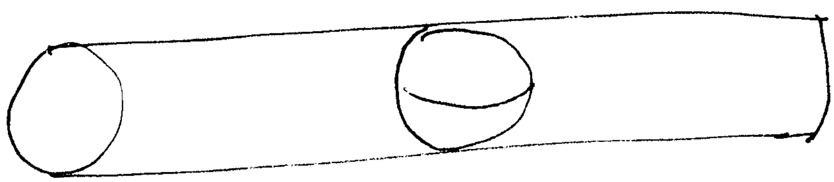
Lecture 6

Linear Superposition and Odds and Ends.

<u>Law of Linear Superposition</u> - The total field at point P is the sum of the fields of the individual charges at P

$$\vec{E}_p = \sum_i \vec{E}_{ip}$$

Consider a long tube of uniform charge density ρ and radius a with a spherical cavity of radius a.



Compute field everywhere.

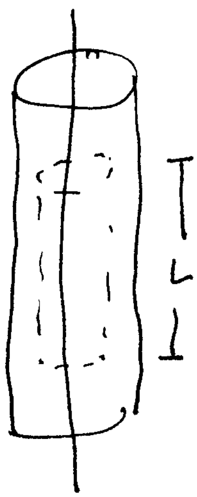
$$\vec{E} = \int \frac{\rho}{4\pi\varepsilon_0} \frac{\hat{r}''}{r''^2} \, dv$$

<u>Ouch</u>!

View system as a superposition of a tube with $+\rho$ and a spherical volume charge with $-\rho$.

$$\vec{E}(\vec{r}) = \vec{E}_{tube} + \vec{E}_{sphere}$$

Pick center of the sphere as the origin and the axis of the tube as the z-axis.



~~<u>Field of the Tube</u> Apply Gauss' law to a co-axial cylindrical of radius r and length L~~

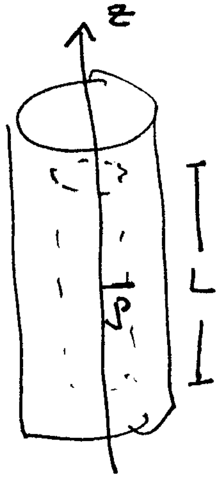
If $r<a$, $Q_{enc} = 2\pi r L \rho$

$$\Phi = 2\pi r L E = \frac{Q_{enc}}{\varepsilon_0} = \pi r^2 L \rho$$

$$\vec{E} = \frac{\rho r}{2\varepsilon_0}\hat{\rho}$$

Using ρ as the volume charge density here introduces severe notational problems, let $\rho \to \gamma$ and reserve ρ for cylindrical coordinate.

<u>Field of Tube</u> - Apply Gauss' law to a co-axial cylindrical Gaussian surface of radius ρ and length L.



$$\Phi = 2\pi\rho L E = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow E(\rho) = \frac{Q_{enc}}{2\pi\rho L \varepsilon_0}$$

If $\rho < a$, $Q_{enc} = \pi \rho^2 L \gamma$

$$\vec{E}(\rho) = \frac{\pi \rho^2 \gamma}{2\pi\rho\varepsilon_0}\hat{\rho} = \frac{\gamma\rho}{2\varepsilon_0}\hat{\rho}$$

If $\rho \geq a$, $Q_{enc} = \pi a^2 L \gamma$

$$\vec{E}(\rho) = \frac{\pi a^2 \gamma}{2\pi\varepsilon_0\rho}\hat{\rho} = \frac{a^2\gamma}{2\varepsilon_0\rho}\hat{\rho}$$

Likewise for the sphere,

$$\vec{E} = \frac{-r\gamma}{3\varepsilon_0}\,\hat{r} \qquad r<a$$

$$\vec{E} = -\frac{\frac{4}{3}\pi a^3\gamma}{4\pi\varepsilon_0 r^2}\,\hat{r} = -\frac{a^3\gamma}{3\varepsilon_0 r^2}\,\hat{r} \qquad r>a$$

From these four fields we can construct the total field

$\rho \geq a$
$$\vec{E} = \frac{a^2\gamma}{2\varepsilon_0\rho}\,\hat{\rho} - \frac{a^3\gamma}{3\varepsilon_0 r^2}\,\hat{r}$$

$r<a$
$$\vec{E} = \frac{\gamma\rho}{2\varepsilon_0}\,\hat{\rho} - \frac{r\gamma}{3\varepsilon_0}\,\hat{r}$$

$r>a,\ \rho<a$
$$\vec{E} = -\frac{a^3\gamma}{3\varepsilon_0 r^2}\,\hat{r} + \frac{\gamma\rho}{2\varepsilon_0}\,\hat{\rho}$$

Lecture 6 - Dipoles

So we can do anything

Linear Charges (λ)

$$\vec{E} = \int \frac{\lambda \, d\vec{l} \, \hat{r}''}{4\pi\varepsilon_0 r''^2} \qquad V = \int \frac{\lambda \, dl}{4\pi\varepsilon_0 r''}$$

Surface Charges (σ)

$$\vec{E} = \int \frac{\sigma \, da \, \hat{r}''}{4\pi\varepsilon_0 r''^2} \qquad V = \int \frac{\sigma \, da}{4\pi\varepsilon_0 r''}$$

Volume Charges (ρ)

$$\vec{E} = \int \frac{\rho \, dv \, \hat{r}''}{4\pi\varepsilon_0 r''^2} \qquad V = \int \frac{\rho \, dv}{4\pi\varepsilon_0 r''}$$

Green's Functions

The potential presents us with a fairly general problem. We have a set of "sources" $\rho(x)$ and a differential equation $L u(x) = \rho(x)$ that defines how the sources act. Our job is to find $u(x)$.

It would be nice to write $u(x)$ as a sum over the sources

$$u(x) = \int G(x,s)\,\rho(s)\,ds$$

where $G(x,s)$ is some function that "propagates" the effect of a source at s to the point x.

The function $u(x)$ must still solve the equation so

$$L u(x) = L \int G(x,s)\,\rho(s)\,ds$$

$$= \int \left[L\, G(x,s) \right] \rho(s)\,ds$$

because L acts on x not s.

evidently $L\, G(x,s) = \delta(x-s)$

$G(x,s)$ is called the Green's function of the operator L (the propogator in particle physiss. The Green's function are the lines in the Feynman diagrams).

Our problem is to solve
$$\nabla \cdot \vec{E} = \rho/\varepsilon_0$$
$$\nabla \times \vec{E} = 0$$

We can solve the second equation if $\vec{E} = -\nabla V$ where V is some function. This turns the first equation into
$$\nabla \cdot (-\nabla V) = \rho/\varepsilon_0$$
or
$$\nabla^2 V = -\rho/\varepsilon_0 \quad \text{(Poisson's Eqn)}$$

Solve $-\varepsilon_0 \nabla^2 V = \rho$ $\Rightarrow$ We want the Green's function of $L = -\varepsilon_0 \nabla^2$

Let's drop the ε_0, since the Green's function for the Laplacian is a pretty general problem.

$$-\nabla^2 (G(\vec{r}, \vec{r}')) = \delta(\vec{r} - \vec{r}')$$

Note ∇ acts on x, y, z not x', y', z'

Looking back, we have

$$-\nabla^2\left(\frac{1}{r''}\right) = 4\pi\,\delta(\vec{r}-\vec{r}'')$$

$$G(\vec{r},\vec{r}\,') = -\frac{1}{4\pi r''} =$$

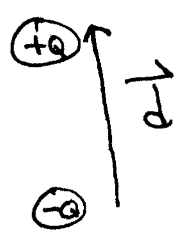
$$r'' = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$$

Dipole Moment

There is a further method of specifying the charge of a system, by the dipole moment per unit volume.

Dipole System - Zero charge but non-zero separation of charge.

Dipole Moment $\quad \vec{P} = Q\,\vec{d}$



Polarization Vector ($\vec{P}$) - Dipole moment per unit volume.

point charge

Dipole Potential (V_{dip}) $= \dfrac{d\left(V(z - d/2) - V(z + d/2)\right)}{d}$

$$= \frac{qd}{4\pi\varepsilon_0}\left(\frac{1/(x^2+y^2+(z-d/2)^2)^{1/2} - \cdots}{d}\right).$$

$$V_{dip} = \frac{qd}{4\pi\varepsilon_0}\left(-\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right) \qquad \lim_{d\to 0}$$

$qd = p$

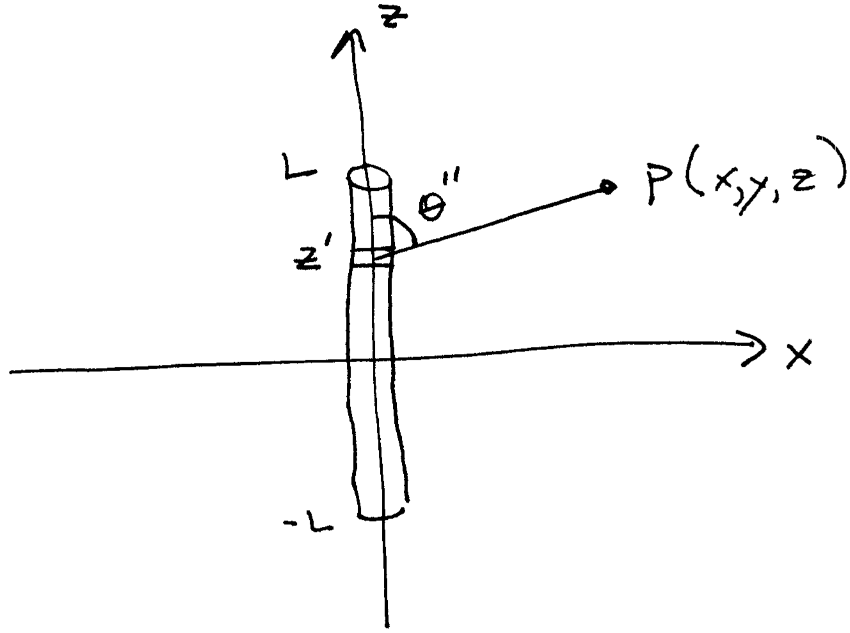
$$V_{dip} = \frac{p}{4\pi\varepsilon_0}\left(\frac{z}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right)$$

$$= \frac{p}{4\pi\varepsilon_0}\,\frac{\cos\theta}{r^2} = \frac{k\,p\cos\theta}{r^2}$$

$$\boxed{V_{dip} = \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}}$$

Calculation of field left as hwk problem.

$\underline{Ex}$ Compute the field of a thin pencil with polarization $\vec{P} = P_0 \hat{z}$ and area A occupying $-L$ to L



$$\vec{r}'' = \vec{r} - \vec{r}' = (x,y,z) - (0,0,z')$$

$$= (x, y, z-z')$$

$$\cos\theta'' = \hat{z}\cdot\hat{r}'' = \frac{z-z'}{\sqrt{x^2+y^2+(z-z')^2}}$$

$$V = \int_{-L}^{L} \frac{k\,dp\cos\theta''}{\left(\sqrt{x^2+y^2+(z-z')^2}\right)^2} \qquad dp = AP_0\,dz'$$

$$= kAP_0 \int_{-L}^{L} \frac{(z-z')\,dz'}{(x^2+y^2+(z-z')^2)^{3/2}}$$

$$V_{d.} = kAP_0 \left(\frac{1}{(x^2+y^2+(z-L)^2)^{1/2}} - \frac{1}{((x^2+y^2+(z+L)^2)^{1/2}} \right)$$

> $int\left(\dfrac{(z-zp)}{\left(x^2+y^2+(z-zp)^2\right)^{\left(\frac{3}{2}\right)}}, zp\right);$

$$\frac{1}{\sqrt{x^2+y^2+z^2-2\,z\,zp+zp^2}} \tag{1}$$

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Electric Force

$$\vec{F} = Q\vec{E}$$

$$\vec{F} = \int_C \vec{E}\lambda\, dl \qquad \vec{F} = \int_S \vec{E}\sigma\, da$$

$$\vec{F} = \int_V \vec{E}\rho\, dv$$

Ex Force of $q_1 = 3nC$ on $q_2 = 5nC$

$$\vec{r}_1 = (3cm, 0, 0) \qquad \vec{r}_2 = (0, 5cm, 0)$$

$$\vec{F}_{12} = \frac{k q_1 q_2}{r''^2}\hat{r}''$$

$$\vec{r}'' = \vec{r}_2 - \vec{r}_1 = (-3cm, 5cm, 0)$$

$$r'' = \sqrt{(3cm)^2 + (5cm)^2}$$

$$= \sqrt{34}\,cm$$

$$\hat{r}'' = \left(\frac{-3}{\sqrt{34}}, \frac{5}{\sqrt{34}}, 0\right)$$