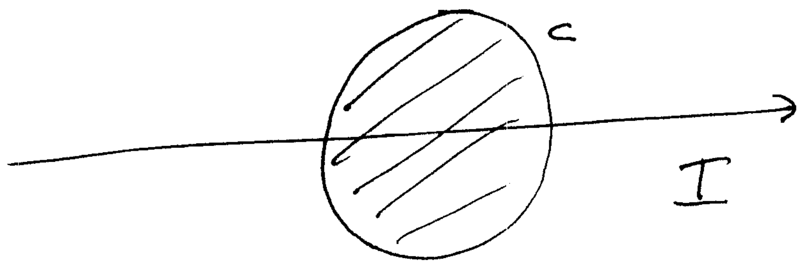
Current

Current (I) - Charge per unit time flowing through the surface S bounded by curve C.



We can generalize I to the vector $\vec{I}$ which points in the direction of current. You can think of $\vec{I}$ as a line current, analogous to a line charge.

In lecture, I said all charges were volume charges and that surface charges and line charges were specific approximations. All currents are volume currents, so let's recover the current density $\vec{J}$.

<u>Current Density ($\vec{J}$)</u> - The current per unit area (or charge per unit area per time) flowing in direction $\hat{n}$.

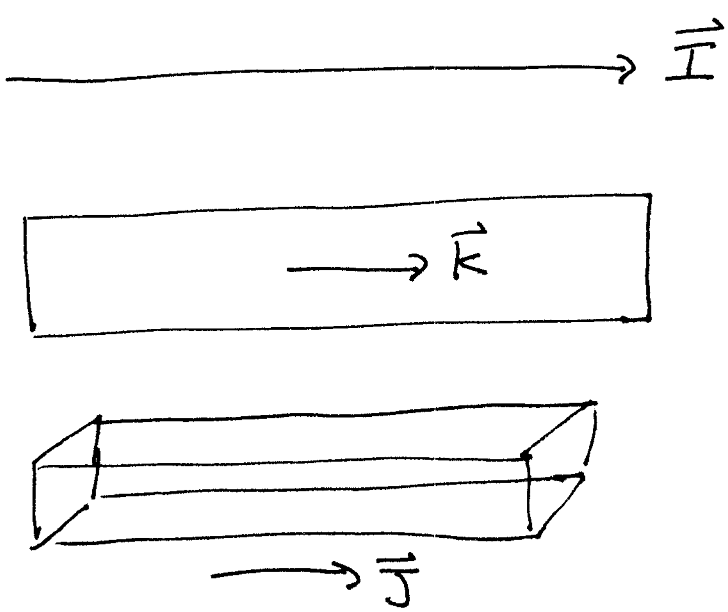
$\Rightarrow$ Let $\hat{n}$ be the direction of flow.

$\Rightarrow$ Let da be an infinitesimal surface with normal $\hat{n}$.

$\Rightarrow$ The magnitude of the current density is the current dI through the surface per da

$$\Rightarrow \quad \vec{J} = \frac{dI}{da}\hat{n} \quad \text{or} \quad \frac{d\vec{I}}{da}$$

<u>Surface Current Density ($\vec{K}$)</u> - Current per unit length flowing through an area da.



Now, we can work backward

$$I = \int_S \vec{J} \cdot d\vec{a}$$

$$= \int_C \vec{K} \cdot d\vec{l}$$

Conservation of Charge - The charge flowing out of a volume per unit time must equal the - time rate of change of the total charge in the volume.

$$I_{out} = \int_S \vec{J} \cdot d\vec{a} = -\frac{\partial}{\partial t} \int_V \rho \, dv$$

$$= \int_V (\nabla \cdot \vec{J}) dv = -\int_V \frac{\partial \rho}{\partial t} dv$$

for all volumes V

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Continuity eqn for charge.

Normally, we think of current as something that flows in a wire. We can also make current by moving charged objects.

$\vec{J} = \rho \vec{v}$ where a charge density ρ moves with velocity $\vec{v}$.

$\vec{K} = \sigma \vec{v}$ where a surface charge σ moves with velocity $\vec{v}$ in the plane of the charge.

$\vec{I} = \lambda \vec{v}$ where a line charge λ moves in the direction of the line.

Note, the restrictions are required to produce currents that do not change with time.

The Biot-Savart law applies to static currents $\Rightarrow$ $\frac{\partial \rho}{\partial t} = 0$

$$\Rightarrow \nabla \cdot \vec{J} = 0$$

__Ex__ Griffith's 5.6 Let a disk carry a constant current density σ and rotate at angular velocity ω about its axis. Calculate the current.

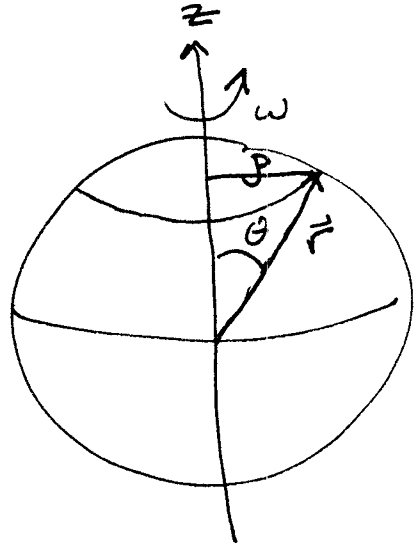
__Sln__ The angular velocity $\omega = \frac{d\theta}{dt}$. This means the velocity of a point on the disk a distance r from the center is

$$\vec{v} = r\frac{d\theta}{dt}\hat{\phi} = r\omega\hat{\phi}$$

and the current density is

$$\vec{K} = r\omega\sigma\hat{\phi} = \sigma\vec{v}$$

Now, let's spin a volume charge with charge density γ.



$$\omega = \frac{d\phi}{dt}$$

$$\vec{J} = \gamma \vec{v} \qquad \rho = r \sin\theta$$

$$\vec{v} = \rho \frac{d\phi}{dt} \hat{\phi} = r\omega \sin\theta \hat{\phi}$$

Dimensionally, we can also write $\vec{I} = \sum q_i \vec{v}$ but this does not yield a quasi-static current.

Each of these new descriptions of current can be used as a source in the Biot-Savart law.

<u>Biot Savart</u>

$$\vec{B} = \frac{\mu_0}{4\pi}\int_c \frac{I\, d\vec{l}' \times \hat{r}''}{r''^2} = \frac{\mu_0}{4\pi}\int_c \frac{\vec{I} \times \hat{r}''}{r''^2}\, dl'$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int_S \frac{\vec{K} \times \hat{r}''}{r''^2}\, da$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(\vec{r}') \times \hat{r}''}{r''^2}\, dv$$

The last form is the most general form. The other versions can be derived as appropriate approximations of this form.

Also, we can write

$$\vec{B} = \frac{\mu_0}{4\pi}\, q\, \frac{\vec{v} \times \hat{r}''}{r''^2}$$

for an isolated charge. This however is a severe approximation and is only good $v \ll c$.

Let's examine $\nabla \cdot \vec{B}$ and $\nabla \times \vec{B}$.

Note, $\vec{B}(\vec{r})$ is a function of x, y, z as is $\nabla \cdot$ and $\nabla \times$.

$$\nabla \cdot \vec{B} = \frac{\mu_0}{4\pi} \int_V \nabla \cdot \left(\frac{\vec{J}(\vec{r}\,') \times \hat{r}''}{r''^2} \right) dv'$$

where the integration is taken over x', y', z'

and $\vec{r}'' = (x-x', y-y', z-z')$

We have a tripple product, so time for those identities.

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

$$\nabla \cdot \left(\frac{\vec{J} \times \hat{r}''}{r''^2} \right) = \nabla \cdot \left(\vec{J} \times \left(\frac{\hat{r}''}{r''^2} \right) \right)$$

$$= \left(\frac{\hat{r}''}{r''^2} \right) \cdot (\nabla \times \vec{J}) - \vec{J} \cdot \left(\nabla \times \left(\frac{\hat{r}''}{r''^2} \right) \right)$$

$\nabla \times \vec{J}(\vec{r}\,') = 0$ because $\vec{r}\,'$ does not depend on x, y, z.

$\nabla \times \left(\frac{\hat{r}''}{r''^2} \right) = 0$ because this is just the radial part of the point charge field and we already know $\nabla \times \vec{E} = 0$

$\Rightarrow \nabla \cdot \vec{B} = 0$

<u>No Magnetic Monopoles</u> – Second Maxwell Eqn

- $\nabla \cdot \vec{B} = 0 \iff \int_S \vec{B} \cdot d\vec{a} = 0 \equiv \Phi_m$
- Total magnetic flux out of a closed surface is zero.
- No isolated magnetic charge.

Now, what is the curl of $\vec{B}$

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \int_V \nabla \times \left(\vec{J}(\vec{r}\,') \times \frac{\hat{r}''}{r''^2} \right) dv'$$

More identities

$$\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B} + \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A})$$

$$\frac{4\pi}{\mu_0} \nabla \times \vec{B} = \int_V \left[\left(\frac{\hat{r}''}{r''^2} \cdot \nabla \right) \vec{J} - (\vec{J} \cdot \nabla) \frac{\hat{r}''}{r''^2} + \vec{J}\left(\nabla \cdot \left(\frac{\hat{r}''}{r''^2} \right) \right) - \frac{\hat{r}''}{r''^2} (\nabla \cdot \vec{J}) \right] dv'$$

$\nabla \cdot \vec{J} = 0$ and $\left(\frac{\hat{r}''}{r''^2} \cdot \nabla \right) \vec{J} = 0$ because $\vec{J}$ depends only on x', y', and z' not x, y, z

$$\nabla \cdot \left(\frac{\hat{r}''}{r''^2} \right) = 4\pi \delta (\vec{r}'')$$

So its piece of the integral is

$$\frac{\mu_0}{4\pi} \int_V \vec{J}(\vec{r}\,') \, \nabla \cdot \left(\frac{\hat{r}''}{r''^2} \right) dv'$$

$$= \frac{\mu_0}{4\pi} \int_V \vec{J}(\vec{r}\,') \, \delta(r'') \, 4\pi \, dv'$$

$$= \mu_0 \vec{J}(\vec{r})$$

Last piece of the integral

$$-\frac{\mu_0}{4\pi} \int \left(\vec{J} \cdot \nabla \right) \frac{\hat{r}''}{r''^2} \, dv'$$

We would like to us $\nabla \cdot \vec{J} = 0$ for static currents or move accurately since $\vec{J}(\vec{r}\,')$, $\nabla' \cdot J(\vec{r}\,') = 0$

where $$\nabla' = \hat{x} \frac{\partial}{\partial x'} + \hat{y} \frac{\partial}{\partial y'} + \hat{z} \frac{\partial}{\partial z'}$$

Lets do this component by component

<u>x- component</u>

$$\frac{\hat{r}''}{r''^2} = \frac{\vec{r}''}{r''^3} = \left(\frac{x-x'}{r''^3}, \frac{y-y'}{r''^3}, \frac{z-z'}{r''^3} \right)$$

so the x- component is

$$(J \cdot \nabla) \frac{x-x'}{r''^3} = -(\vec{J} \cdot \nabla') \frac{x-x'}{r''^3}$$

because x only appears in the combination

$x-x'$

Now, we need to commute $\vec{J}$ and ∇' properly, so more identities

$$\nabla \cdot (f \vec{A}) = f(\nabla \cdot \vec{A}) + (\vec{A} \cdot \nabla) f$$

$$(\vec{J} \cdot \nabla') \frac{x-x'}{r''^3} = \nabla' \cdot \left(\vec{J} \frac{x-x'}{r''^3} \right)$$

$$- \frac{x-x'}{r''^3} \nabla' \cdot \vec{J}$$

The last term is zero. So we are left with

$$-\frac{\mu_0}{4\pi}\int_V \nabla'\cdot\left(J\,\frac{x-x'}{r''^3}\right)dv'$$

$$= -\frac{\mu_0}{4\pi}\int_S \left(\vec{J}\,\frac{x-x'}{r''^3}\right)\cdot d\vec{a}$$

which is zero if we make S large enough to enclose all current

<u>Putting Everything Back Together</u>

<u>Ampere's Law</u> $\qquad \nabla\times\vec{B} = \mu_0\vec{J}$

(static currents)

Consider the current through surface S,

$$\mu_0 I = \mu_0\int_S \vec{J}\cdot d\vec{a} = \int_S (\nabla\times\vec{B})\cdot d\vec{a}$$

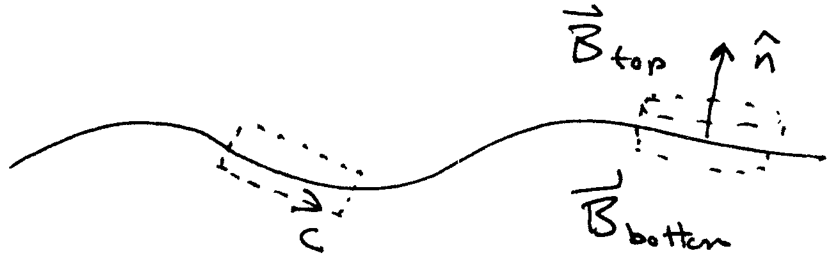
Stoke's Thm

$$\int_S \nabla\times\vec{B}\cdot d\vec{a} = \oint_C \vec{B}\cdot d\vec{l}$$

Ampere's Law Integral Form The integral of the magnetic field around a closed ~~surface~~ loop C is proportional to the current through the surface S bounded by C.

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

Magnetostatic Boundary Conditions



Apply $\nabla \cdot \vec{B}$ to a short cylinder at the surface

$$\Phi_m = \left(\vec{B}_{top} \cdot \hat{n} A\right) + \left(-\vec{B}_{bottom} \cdot \hat{n} A\right) = 0$$

$\Rightarrow$ The normal component of the magnetic field is ~~continuous~~ continuous.

Apply $\nabla \times \vec{B}$ to a curve C at the surface

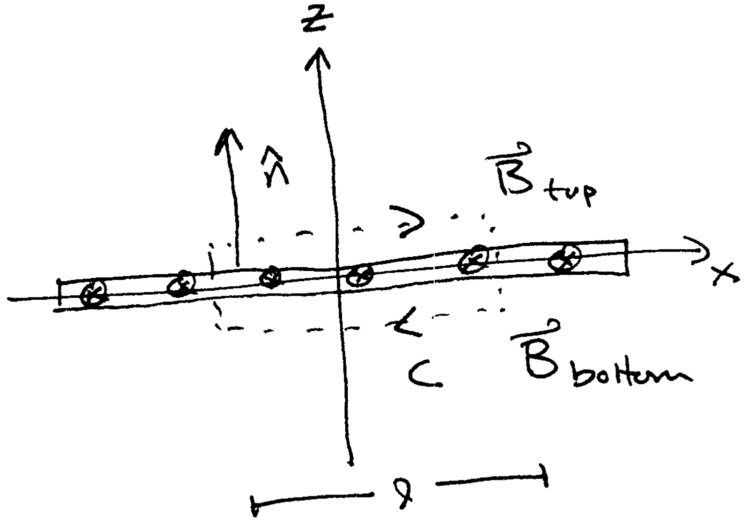
$$\int_C \vec{B} \cdot d\vec{l} = \vec{B}_{bottom} \hat{t}\, l - \vec{B}_{top} \cdot \hat{t}\, l$$

$$= \mu_0 \int_S \vec{J} \cdot d\vec{a} = \mu_0 \vec{K} \cdot \hat{a}$$

where $\vec{t}$ is the tangent and $\hat{a}$ is the surface normal to the curve bounded by C.

$\Rightarrow$ The tangential component changes proportional to the surface current.

Let's repackage this into a somewhat nicer form



y into page

Let a surface current $\vec{K} = (0, K_y, 0)$

The direction of C is chosen to make $\vec{K}$ a positive current by the RHR.

$$B_{z\,top} = B_{z\,bottom} \qquad (\nabla \cdot \vec{B} = 0)$$

$$B_{top\,x}\, l - B_{bottom\,x}\, l = \mu_0 K_y l$$

$$B_{top\,x} - B_{bottom\,x} = \mu_0 K_y$$

Now consider $\vec{K} \times \hat{n} = K_y \hat{y} \times \hat{z} =$

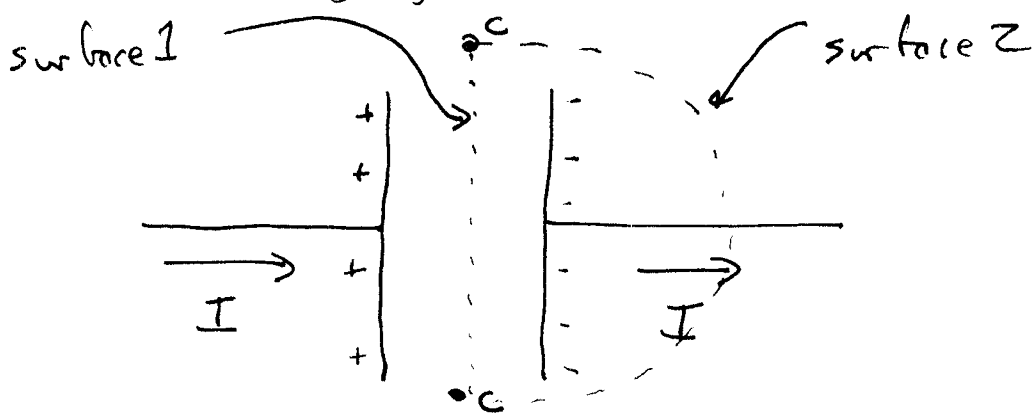
$$= K_y \hat{x}$$

We can re-write both boundary conditions as

$$\vec{B}_{top} - \vec{B}_{bottom} = \mu_0 (\vec{K} \times \hat{n})$$

Ampere's Law is not complete

Consider a charging capacitor



$$|\vec{E}| = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A\varepsilon_0}$$

We can apply Ampere's law to ANY surface bounded by C

<u>surface 1</u> $\quad I_{enc} = 0 \implies \oint_C \vec{B}\cdot d\vec{l} = 0$

<u>surface 2</u> $\quad I_{enc} = I \implies \oint_C \vec{B}\cdot d\vec{l} = \mu_0 I$

We need something with the dimensions of current that is zero for surface 2 but equals the current for surface 1. This could be accomplished if we used the electric field.

Consider the electric flux Φ through surface 1.

$$\Phi = EA = \frac{\sigma}{\varepsilon_0} A = \frac{Q}{\varepsilon_0}$$

$$\frac{d\Phi}{dt} = \frac{1}{\varepsilon_0} \frac{dQ}{dt} = \frac{I}{\varepsilon_0}$$

$$I_d = \varepsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot d\vec{a}$$

Displacement Current (I_d) – A "~~fictitious~~" quantity with the units of current needed to complete Ampere's Law.

$$I_d = \varepsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot d\vec{a}$$

Complete Ampere's Law

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot d\vec{a}$$

or

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \vec{J}_d$$

Where $\vec{J}_d$ is the displacement current density

$$\vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \text{Ampere's Law}}$$

$\Rightarrow$ Changing electric field produces a magnetic field.

Let's play with this some more. Another way to see Ampere's is incomplete.

$$\nabla \cdot (\nabla \times \vec{B}) = \mu_0 \nabla \cdot \vec{J}$$ unfixed Amperes

- $\nabla \cdot (\nabla \times \vec{A}) = 0$ for all vectors.
- $\Rightarrow$ $\nabla \cdot \vec{J} = 0$ which is true steady currents but is certainly not true for all currents.

The continuity equation is true for all currents

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Gauss' Law is true perod. $\nabla \cdot \vec{E} = \rho/\varepsilon_0$

$$\rho = \varepsilon_0 \nabla \cdot \vec{E}$$

So

$$\nabla \cdot \vec{J} + \varepsilon_0 \nabla \cdot \frac{\partial \vec{E}}{\partial t} = 0 \quad \text{always}$$

Thrs suggests if we want $\nabla \cdot \vec{J} = 0$ always we augment the current with an additional term $\vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

Let's accept Ampere's Law and see what we can dig out of it.

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot (\nabla \times \vec{B}) = 0 = \mu_0 \nabla \cdot \vec{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E}$$

But from Gauss' Law $\nabla \cdot \vec{E} = \rho/\varepsilon_0$

$$0 = \mu_0 \nabla \cdot \vec{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \rho/\varepsilon_0$$

$$0 = \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t}$$

$\Rightarrow$ Conservation of charge is contained in Gauss + Ampere.