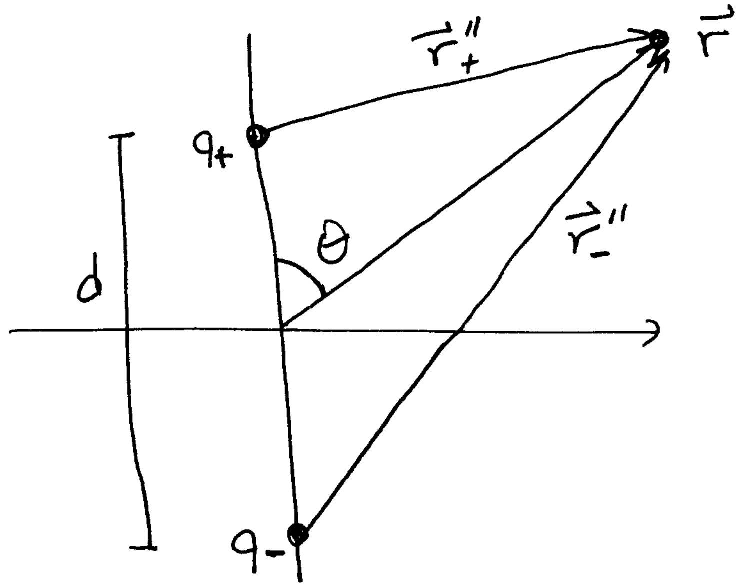
Electric Dipoles

Consider a simple model of a dipole,



The potential at the point $\vec{r}$ is

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r''_+} - \frac{1}{r''_-}\right)$$

where $q_+ = +q$ $\qquad q_- = -q$

Apply law of cosines,

$$r''^2_\pm = r^2 + \left(\frac{d}{2}\right)^2 \mp r\,d\cos\theta = r^2\left(1 \mp \frac{d}{2r}\cos\theta + \frac{d^2}{4r^2}\right)$$

Note $\cos(\theta - \pi) = -\cos\theta$

Keep the first two terms if $r >> d$

$$\frac{1}{r''_{\pm}} \simeq \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2}$$

$\Rightarrow$ binomial expansion

$$\frac{1}{r''_{\pm}} \simeq \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

$$\Rightarrow \frac{1}{r''_{+}} - \frac{1}{r''_{-}} = \frac{1}{r}\left(1 + \frac{d}{2r}\cos\theta\right) - \frac{1}{r}\left(1 - \frac{d}{2r}\cos\theta\right)$$

$$= \frac{d}{r^2}\cos\theta$$

$$V(\vec{r}) \simeq \frac{qd}{4\pi\varepsilon_0}\,\frac{\cos\theta}{r^2}$$

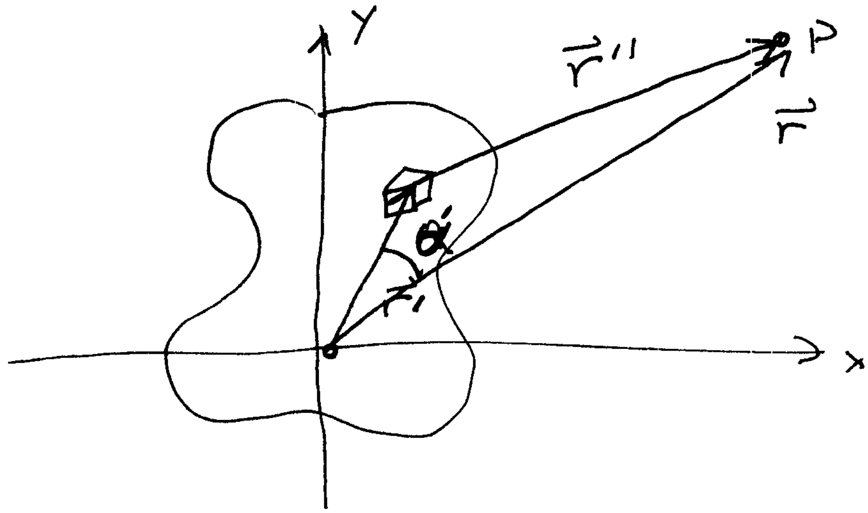
Let the dipole moment vector point from the - charge to the + charge and have magnitude qd

the $\vec{p}\cdot\hat{r} = qd\cos\theta$

$$V(\vec{r}) = \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}$$

Let's ~~thry~~ this for ~~an~~ arbitrary charge distribution $\rho(\vec{r})$

$$V(\vec{r}) = \int \frac{\rho(\vec{r}')\,dv'}{4\pi r''}$$



$\alpha \equiv$ Angle between $\vec{r}$ and $\vec{r}'$

Law of Cosines

$$r''^2 = r'^2 + r^2 - 2rr'\cos\alpha$$

$$= r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]$$

$$= r^2\left[1 + \underbrace{\frac{r'}{r}\left(\frac{r'}{r} - 2\cos\alpha\right)}_{s}\right]$$

where s is small.

$$\frac{1}{r''} = \frac{1}{r}\left(\sqrt{1+s}\right)^{-1}$$

$$= \frac{1}{r}\left(1 - \frac{1}{2}s + \frac{3}{8}s^2 - \frac{5}{16}s^3 + \cdots\right)$$ binomial expansion

$$= \frac{1}{r}\left(1 - \frac{r'}{2r}\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\alpha\right)^3 + \cdots\right)$$

$$= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)\cos\alpha + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\alpha - 1}{2}\right) + \cdots\right]$$

The angle expressions are the Legendre Polynomials, P_n, consult your math handbook.

$$\frac{1}{r''} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\alpha)\, \rho(\vec{r}')\, dv'$$

$$= \frac{1}{4\pi\epsilon_0} \left[\frac{1}{r} \int \rho(\vec{r}')\, dv' + \frac{1}{r^2} \int r' \cos\alpha\, \rho(\vec{r}')\, dv' \right.$$

$$\left. + \frac{1}{r^3} \int (r')^2 \left(\frac{3}{2}\cos^2\alpha - \frac{1}{2} \right) \rho(\vec{r}')\, dv' + \cdots \right.$$

$\Rightarrow$ Multipole expansion of V

Monopole Moment

$$Q_T = \int \rho(\vec{r}')\, dv' = \text{Total charge}$$

$\Rightarrow$ note potential about origin, not center of charge.

Dipole Term

$$V_{d.p}(\vec{r}) = \frac{1}{4\pi\epsilon_0 r^2} \int r' \cos\alpha\, \rho(\vec{r}')\, dv'$$

$$r'\cos\alpha = \vec{r}\,'\cdot\hat{r}$$

$$V_{dip} = \frac{\hat{r}}{4\pi\varepsilon_0 r^2}\cdot\int \vec{r}\,'\rho(\vec{r}\,')\,dv'$$

<u>Dipole Moment</u>

$$\vec{p} \equiv \int \vec{r}\,'\rho(\vec{r}\,')\,dv'$$

$$V_{dip} = \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2} \quad \text{as promised}$$

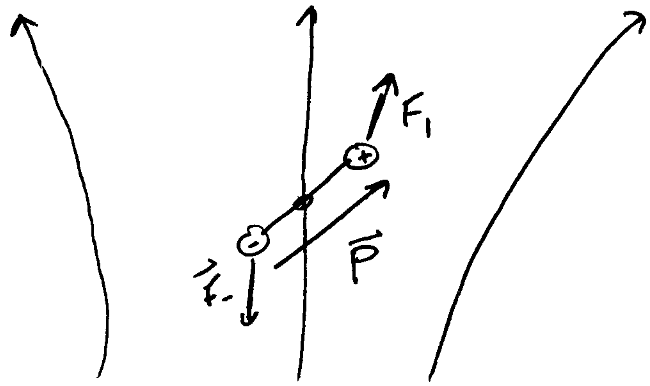
$\Rightarrow$ Dipole moment of point charge, that is not at origin, is not zero.

$\Rightarrow$ If $Q=0$, the dipole moment does not depend on origin.

$\Rightarrow$ You have already calculate dipole electric field

$$\vec{E}_{dip} = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

Electric Dipole Mechanics



Torque $\quad \vec{\tau} = \vec{p} \times \vec{E}$

Potential Energy of Rotation (

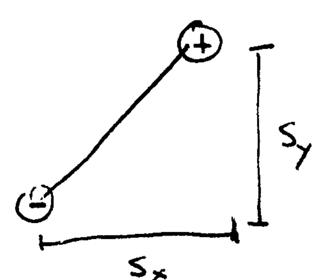
$$U = \int_{\pi/2}^{\pi} \tau \, d\theta = -\vec{p} \cdot \vec{E}$$

$\pi/2 \leftarrow$ zero energy

⇒ Permanent Dipole

Force on Dipole

Examine the figure above, the electric field decreases in strength in the y-direction which implies a net force on the dipole.



$$F_y = Q \Delta E_y = Q \frac{\partial E_y}{\partial y} s_y$$

$$= p_y \frac{\partial E_y}{\partial y}$$

But from our diagram, the electric field also changes in the x and z directions.

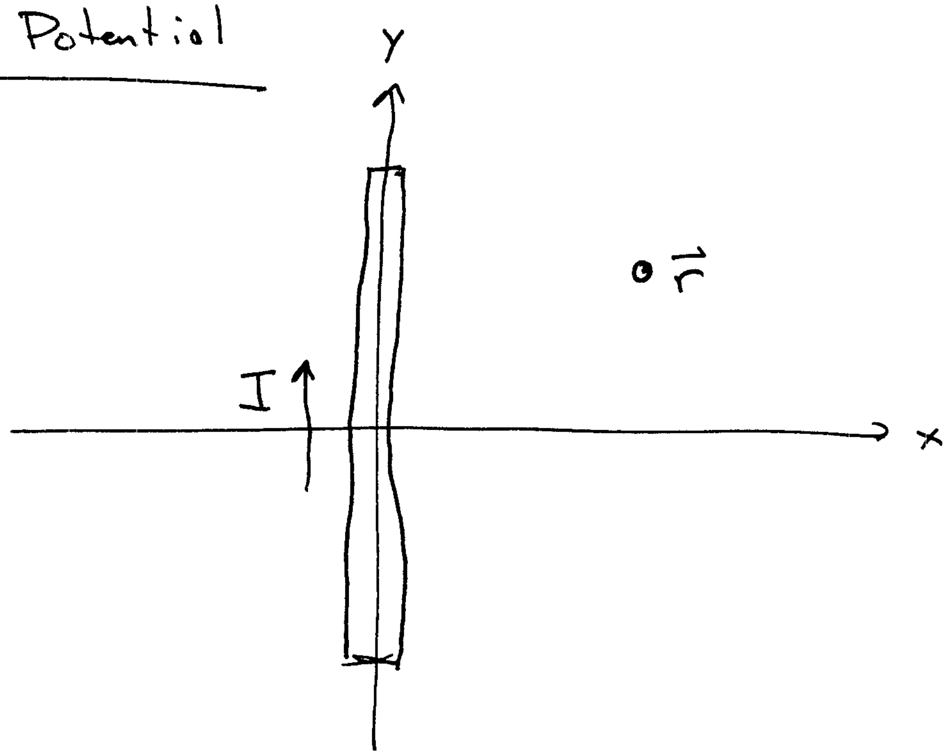
$$F_y = p_x \frac{\partial E_y}{\partial x} + p_y \frac{\partial E_y}{\partial y} + p_z \frac{\partial E_y}{\partial z}$$

or

$$\vec{F} = (\vec{p} \cdot \nabla) \vec{E}$$

If $\nabla \times \vec{E} = 0$ (static), $\vec{F} = \nabla (\vec{p} \cdot \vec{E})$

Vector Potential



$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I}\, dl'}{r''}$$

$$\vec{I} = I\hat{y}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I \hat{y}}{4\pi} \int \frac{dl'}{r''}$$

$\Rightarrow$ The vector potential points in the same direction as the current at all points in space.

$$\vec{r} = (x, y, z)$$

$$\vec{r}' = (0, y', 0) \qquad \vec{r}'' = (x, y-y', z)$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I \hat{y}}{4\pi} \int_{-a/2}^{a/2} \frac{dy'}{\sqrt{x^2+z^2+(y-y')^2}}$$

$$= -\frac{\mu_0 I \hat{y}}{4\pi} \int_{y+a/2}^{y-a/2} \frac{du'}{\sqrt{(x^2+z^2)+u^2}}$$

$$u = y - y' \qquad du = -dy'$$

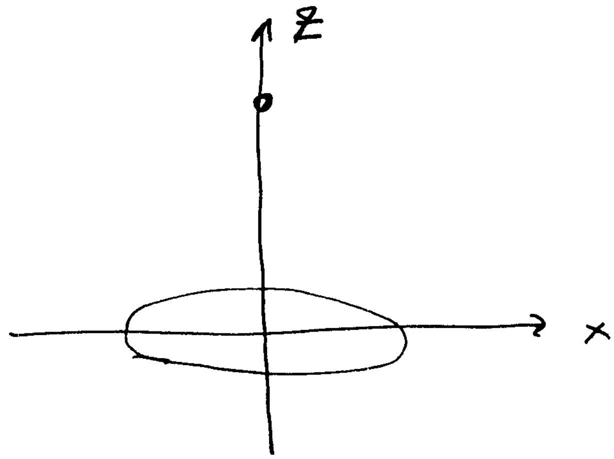
$$\vec{A}(\vec{r}) = \frac{-\mu_0 I \hat{y}}{4\pi} \ln\left(u + \sqrt{u^2 + (x^2+z^2)}\right) \Bigg|_{y+a/2}^{y-a/2}$$

$$= \frac{\mu_0 I \hat{y}}{4\pi} \ln\left(\frac{y+a/2 + \sqrt{(y+a/2)^2+x^2+z^2}}{y-a/2+\sqrt{(y-a/2)^2+x^2+z^2}}\right)$$

Suppose we are on the axis, $y=0$, $z=0$

$$\vec{A}(x,0,0) = \frac{\mu_0 I \hat{y}}{4\pi} \ln\left(\frac{a/2 + \sqrt{(a/2)^2+x^2}}{-a/2+\sqrt{(a/2)^2+x^2}}\right)$$

Try a different system. Vector potential on z-axis of a loop in x-y plane.

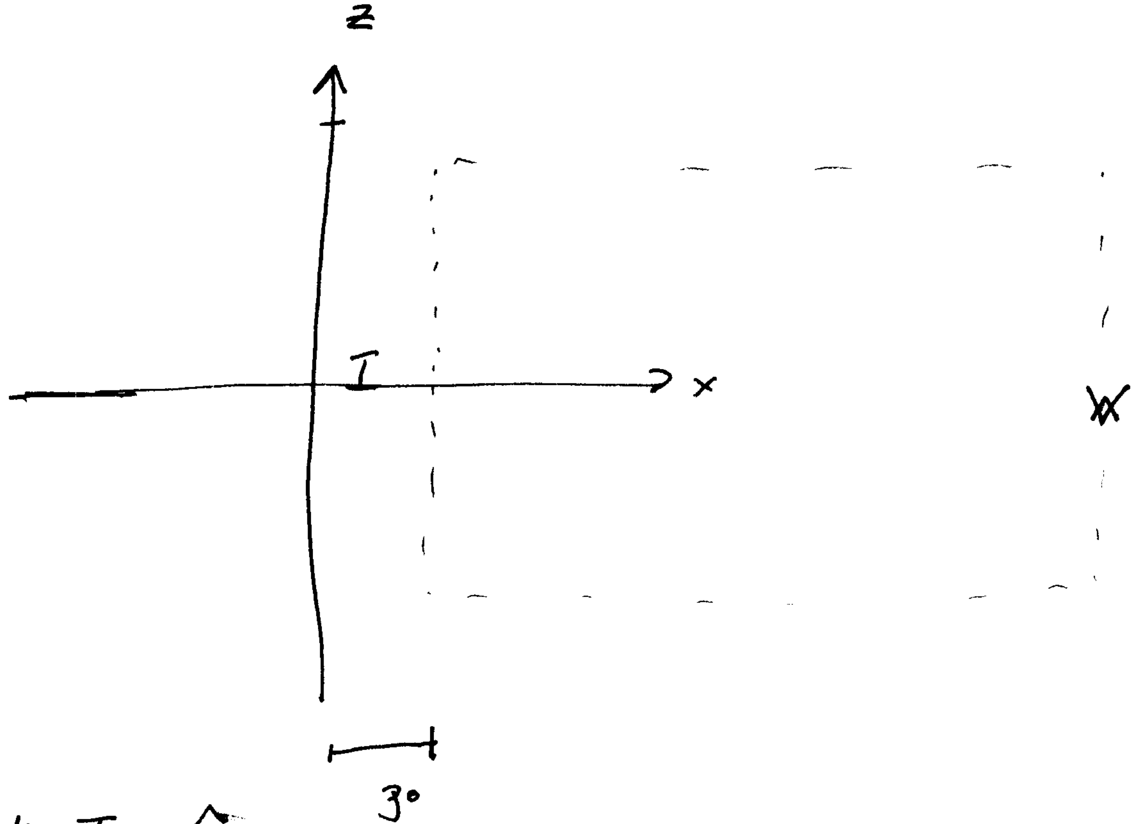


$$\vec{I} = I\hat{\phi}$$

$$\vec{A}(z) = \frac{\mu_0}{4\pi} \oint_c \frac{\vec{I}\, dl'}{r''} = 0$$

But this does <u>not</u> imply the field is zero.

Ex <u>Infinite Wire</u>



$$\vec{B} = \frac{\mu_0 I}{2\pi\rho}\hat{\phi}$$

$$\vec{A} = \int \frac{\mu_0 \vec{I}\, dl'}{4\pi r''} \implies \vec{A} \parallel \hat{z}$$

Both the integral for Φ_m and the definition in terms of $\vec{I}$ diverges.

Try to solve the curl

$$\cancel{\nabla\times\vec{A} = \frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial\rho} - \frac{\partial A_\rho}{\partial\phi}\right)\hat{z} =}$$

Keep only $\hat{\phi}$ term

$$\left(\nabla \times \vec{A}\right)_{\phi} = \left(\frac{\partial A_{\rho}}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) = \frac{\mu_0 I}{2\pi \rho}$$

$$A_{\rho} = 0$$

$$A_z = -\frac{\mu_0 I}{2\pi} \ln\left(\rho/\rho_0\right)$$

where $\vec{A}(\rho_0) \equiv 0$.

$$\vec{A} = -\frac{\mu_0 I}{2\pi} \ln\left(\rho/\rho_0\right) \hat{z}$$

Method II

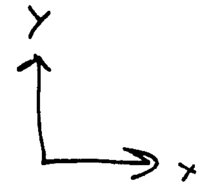
Select $\vec{A}(\rho_0) = 0$

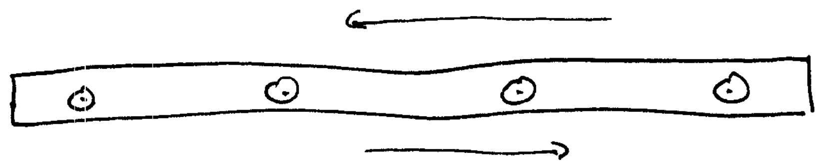
$$\oint \vec{A} \cdot d\vec{l} = \Phi_m = l \int_{\rho_0}^{\rho} \frac{\mu_0 I \, d\rho}{2\pi\rho}$$

$$-Al = \frac{l \mu_0 I}{2\pi} \ln \rho/\rho_0$$

$$\vec{A} = \frac{-\mu_0 I}{2\pi} \ln \rho/\rho_0 \, \hat{z} \qquad \rho > \rho_0$$

Infinite Plane





$\vec{K} = K\hat{z}$

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}\,da'}{r''}$$

$\Rightarrow$ $\vec{A}$ is in $\hat{z}$ direction

$$\vec{B} = \begin{cases} -\mu_0 K\,\hat{x} & y > 0 \\ \mu_0 K\,\hat{x} & y < 0 \end{cases}$$

$$(\nabla \times \vec{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\partial A_z}{\partial y}$$

$$\frac{\partial A_z}{\partial y} = -\mu_0 K$$

$$A_z = -\mu_0 K(y)$$

$$\vec{A} = -\mu_0 K y\,\hat{z}$$

Magnetic Multipole Expansion

The vector potential of a current loop is

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l}'}{r''}$$

$$= \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos\alpha)\, d\vec{l}'$$

where α is the angle between $\vec{r}$ and $\vec{r}\,'$ as before.

Monopole Term

$$\vec{A}_{mono} = \frac{\mu_0 I}{4\pi r} \oint d\vec{l}' = 0$$

Dipole Term

$$\vec{A}_{dipole} = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\alpha \, d\vec{l}'$$

As before, $r' \cos\alpha = \vec{r}' \cdot \hat{r}$

Vector Identity $\oint_C (\hat{r} \cdot \vec{r}\,') d\vec{l}\,' = -\hat{r} \times \int d\vec{a}\,'$

$$= -\hat{r} \times \vec{a}$$

$$= \vec{a} \times \hat{r}$$

Magnetic Dipole Moment

$$\vec{m} = I \vec{a} \qquad \vec{a} = \int_S d\vec{a}$$

RHR - Curl fingers in direction of current, fingers point in direction of field.

Magnetic Dipole Potential

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$$

Magnetic Dipole Field

$$\vec{B}_{dip} = \frac{\mu_0 m}{4\pi r^3} \left(2\cos\theta \, \hat{r} + \sin\theta \, \hat{\theta} \right)$$

Magnetic Dipole Mechanics

Torque (You calculated this)

$$\vec{\tau} = \vec{m} \times \vec{B}$$

Potential Energy

$$U = -\vec{m} \cdot \vec{B}$$

Force (I didn't make you calculate this)

$$\vec{F} = \nabla(\vec{m} \cdot \vec{B})$$

$\Rightarrow$ Difference from electric for is due to lack of monopoles.

Ex Multipole expansion of point charge at $\rho(\vec{r}) = Q\,\delta(\vec{r}-\vec{r}_A)$

$$\vec{r}_A = (a, 0, 0)$$

Monopole

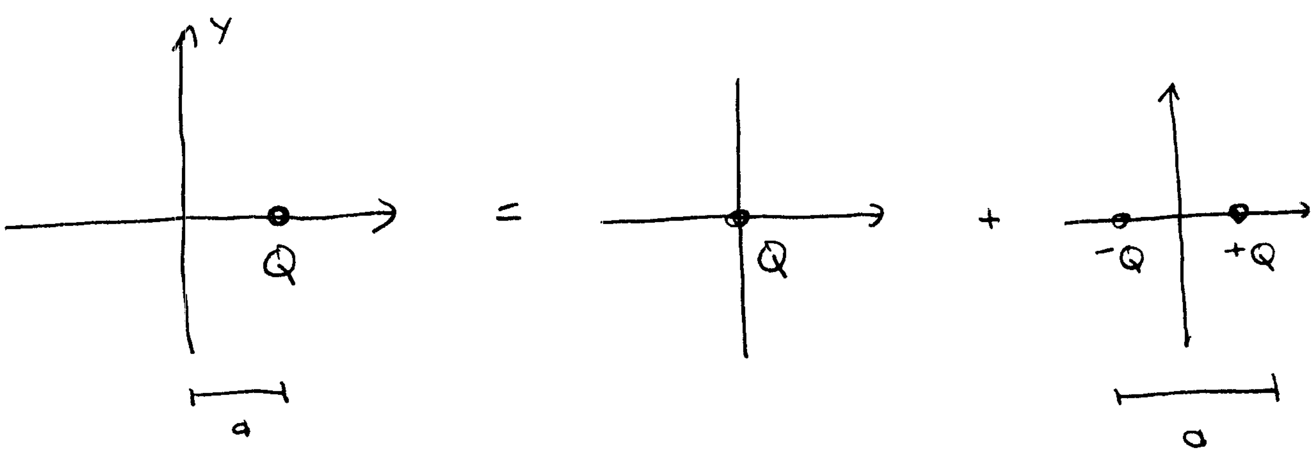
$$Q_m = Q = \int \rho(\vec{r}\,')\,dv'$$

Dipole

$$\vec{p} = \int \vec{r}\,'\rho(\vec{r}\,')\,dv'$$

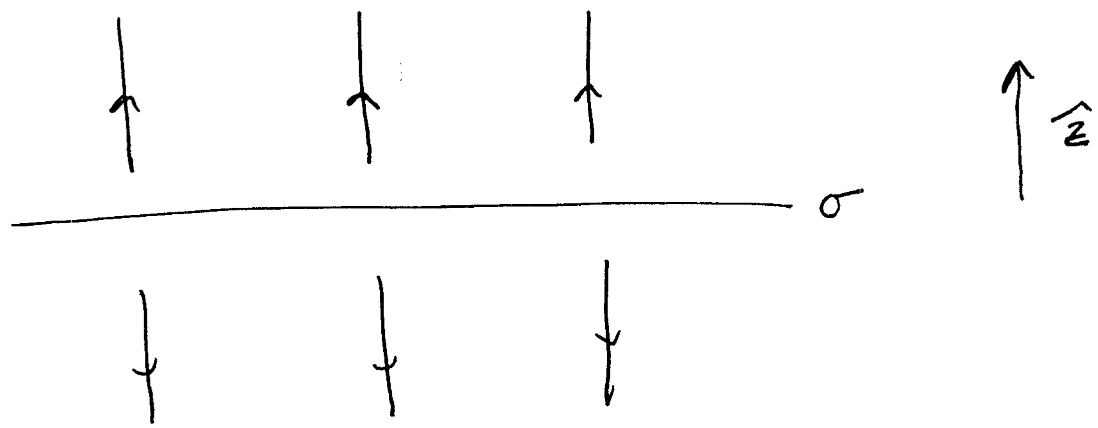
$$= \int \vec{r}\,' Q\,\delta(\vec{r}\,'-\vec{r}_A)\,dv'$$

$$= \vec{r}_A Q$$



+ other terms.

Ex Compute Potential of Infinite Plane with surface charge density σ



$$V(\vec{r}) = \int \frac{\sigma\, da'}{4\pi r'' \varepsilon_0} = \infty$$

$\Rightarrow$ Problem - Choice of $V(\infty) = 0$ gives infinite potential.

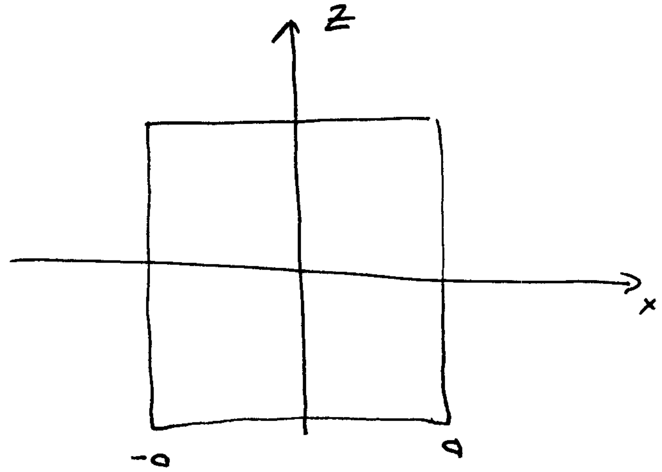
$$-\nabla V = \vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{z} \qquad z > 0$$

$$V = -\frac{\sigma}{2\varepsilon_0} z + C$$

Let $V(0) = 0$, $V(z) = -\dfrac{\sigma}{2\varepsilon_0}|z|$

$\Rightarrow$ Fixed sign by considering work.

Dipole Moment



$\sigma_+ = +\sigma \quad z > 0$

$\sigma_- = -\sigma \quad z < 0$

$$\vec{P} = \int_{-a}^{a} dz \int_{-a}^{a} dx \; \vec{r}\,'\sigma(\vec{r}\,')\,da'$$

$$P_x = \int_{-a}^{a} dz' \int_{-a}^{a} dx'\, x'\,\sigma(\vec{r}\,')\,da' = 0$$ (the z integral is zero)

$$P_z = \int_{-a}^{a} dz' \int_{-a}^{a} dx'\, z'\,\sigma(\vec{r}\,')\,da'$$

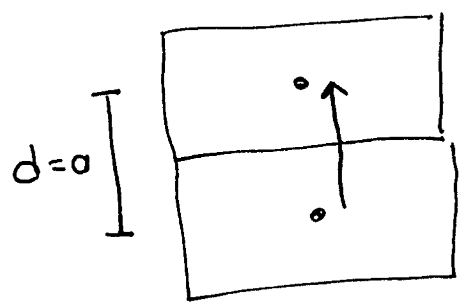
$$= 2\int_{0}^{a} dz' \int_{-a}^{a} dx'\, z'\sigma\, dx'\,dz'$$ since the sign of z' kills the sign change in σ

$$= 2\sigma\left(\frac{a^2}{2}\right)(2a)$$

$$= 2\sigma a^3$$

$$\vec{p} = \frac{\sigma a^3}{2}\,\hat{z}$$

Let's play with it some more.



$Q_+ = 2a^2\sigma$

$Q_- = 2a^2\sigma$

So the dipole moment computed from the center of positive charge to the center of negative charge is

$$\vec{p} = Q_+ d\,\hat{z} = 2a^3\sigma\,\hat{z} \quad \checkmark$$

Now, compute the field at the point $\vec{r}_p = (5a, 0, 5a)$

Use the point dipole approximation,

$$\vec{E}_{dip} = \frac{p}{4\pi\epsilon_0 r^2}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

θ angle from z axis.

$\vec{E}(\vec{r}_p) =$

$\theta = 45°$ $\quad \cos\theta = \frac{1}{\sqrt{2}}$ $\quad \sin\theta = \frac{1}{\sqrt{2}}$

$r = 5\sqrt{2}\, a$

$$\vec{E} = \frac{2a^3\sigma}{4\pi\varepsilon_0 (5\sqrt{2}\, a)^2}\left(\sqrt{2}\,\hat{r} + \frac{1}{\sqrt{2}}\hat{\theta}\right)$$

$$= \frac{\sqrt{2}\, a\sigma}{20\pi\varepsilon_0}\left(\hat{r} + \frac{1}{2}\hat{\theta}\right)$$