Polarization

The polarization ($\vec{P}$) is the dipole moment per unit volume.

If the polarization is uniform, the total dipole moment of an object with volume V is

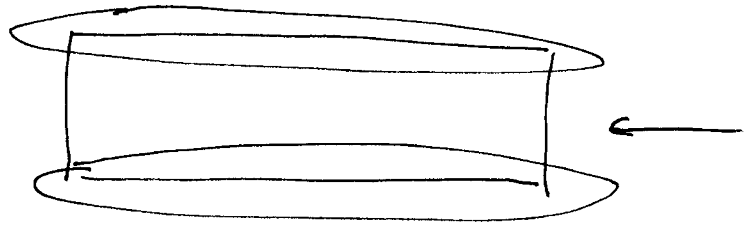
$$\vec{p} = \vec{P} V$$

Since we already know the electric potential of a point dipole

$$V_{dip} = \frac{\vec{p} \cdot \hat{r}}{4\pi\varepsilon_0 r^2}$$

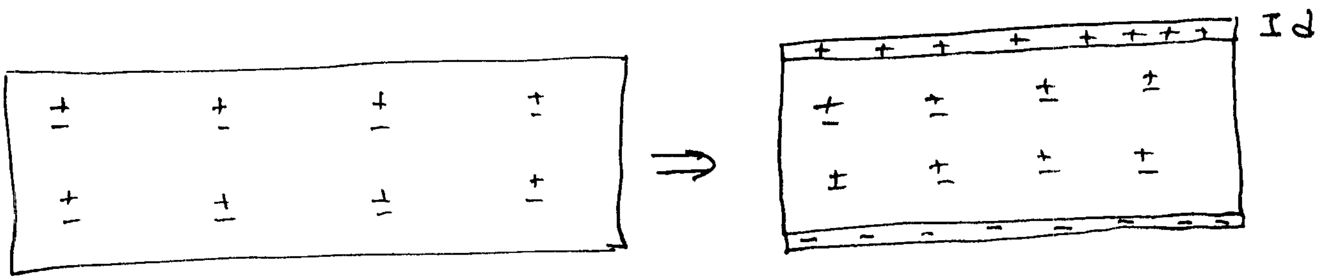
we can immediately get the field ~~far~~ from the object.

Let's look a little nearer the object.



polarization $\vec{P}$

Consider a thin slab with polarization $\vec{P}$. Atomically, $\vec{P}$ results from a displacement of the centers of positive and negative charge in the atoms. We can make a model of this by taking uniform charge densities $\rho_+ = +\rho$ and $\rho_- = -\rho$ and displacing them by d (move + charge up).



This produces a positive surface charge $\sigma_t = \rho_+ d$ on the top and a negative surface charge $\sigma_b = \rho_- d$ on the bottom. If the slab has area A and heigth l the total dipole moment is $p = Qd = A l \rho_+ d$

The polarization is then

$$P = \frac{p}{V} = \rho d$$

and the bound charge density is

$$\sigma_t = P$$

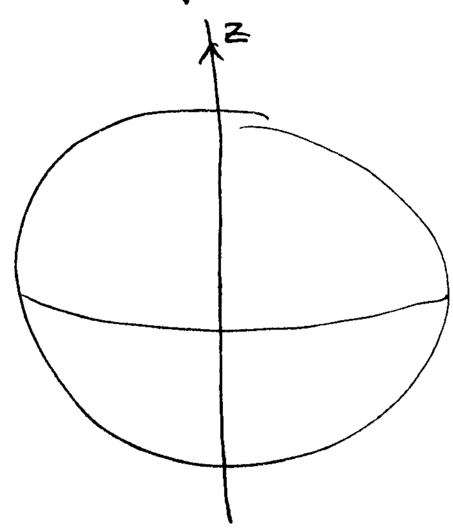
The bound charge produces an electric field

$$\vec{E}_{polar} = \frac{\sigma_t}{\varepsilon_0} = -\frac{\vec{P}}{\varepsilon_0}$$

So far from the edges of an object with uniform polarization $\vec{P}$, the electric field is

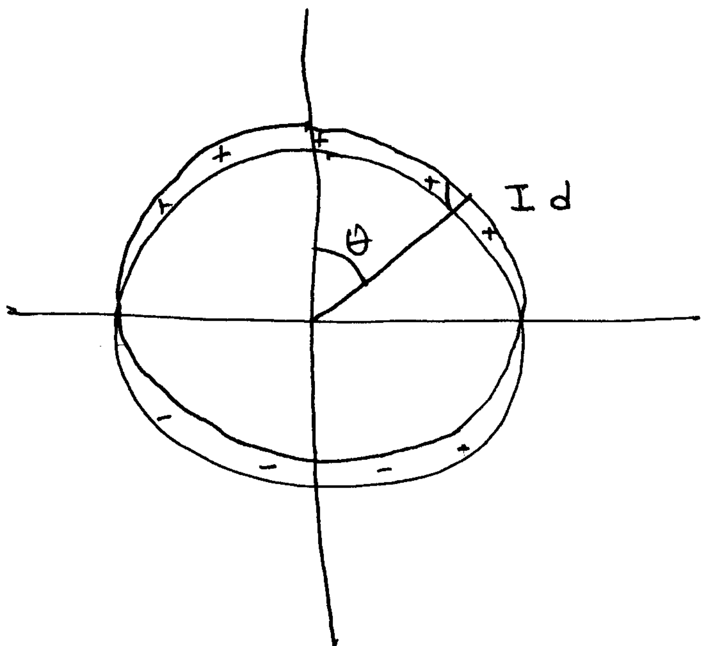
$$\vec{E} = -\frac{\vec{P}}{\varepsilon_0}$$

<u>Case II</u> Uniform spheres with polarization $\vec{P} = P_0\hat{z}$



Pull the same trick. Model the system as spheres with charge density ρ and displace the spheres by d.

The total dipole moment is $p = Qd$. The dipole moment per unit volume is the $P = \frac{p}{V} = \rho d$.



Since the + sphere is displaced d upward everywhere, the thicknes of the boundary layer is $d \cos\theta$ and the surface charge density is $\sigma = \rho d \cos\theta = P\cos\theta$.

Outside a uniform sphere of charge, the field is that of a point charge at the center. So outside the sphere, the potential and field is that of a pure dipole with dipole moment $p = Qd = PV$

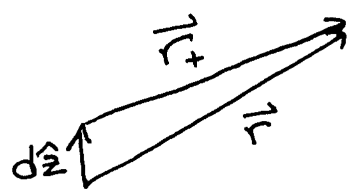
Inside ~~the~~ a sphere with uniform charge density ρ, the field is

$$\vec{E} = \frac{\rho \vec{r}}{3\varepsilon_0}$$

If we displace the positive sphere, the total field is

$$\vec{E} = -\frac{\rho \vec{r}}{3\varepsilon_0} + \frac{\rho \vec{r}_+}{3\varepsilon_0}$$

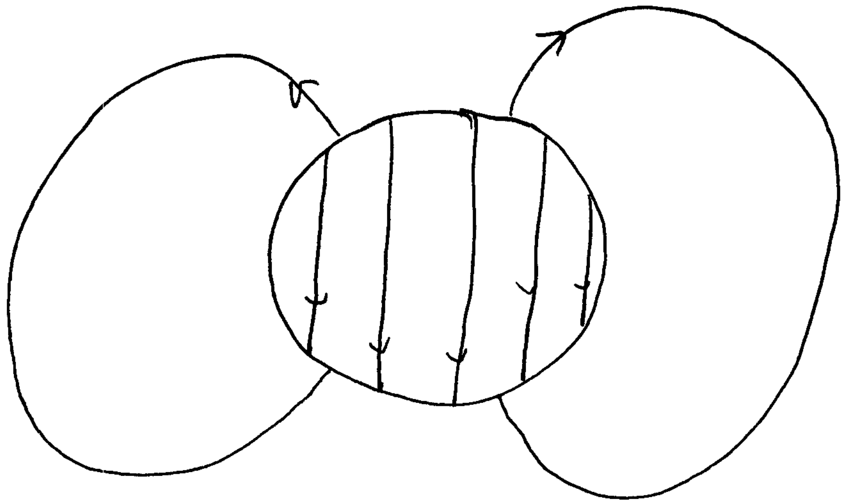
where $\vec{r}_+$ is the vecter from the center of the positive sphere.



$$\vec{r}_+ + d\hat{z} = \vec{r} \qquad \vec{r}_+ = \vec{r} - d\hat{z}$$

$$\vec{E} = -\frac{\rho \vec{r}}{3\varepsilon_0} + \frac{\rho}{3\varepsilon_0}\left(\vec{r} - d\hat{z}\right)$$

$$\vec{E} = \frac{\rho d}{3\varepsilon_0}\hat{z} = -\frac{P}{3\varepsilon_0}\hat{z}$$



For my comfort, let's compute dipole moment of a surface charge density $\sigma = P\cos\theta$

$$\vec{p} = \int \vec{r}\,' P\cos\theta' da'$$

$$da' = (R\,d\theta')(R\sin\theta' d\phi')$$

$$\vec{p} = PR^2 \int \vec{r}\,' \sin\theta' \cos\theta' d\theta' d\phi'$$

From Griffith's

$$\vec{r}\,' = (R\sin\theta'\cos\phi', R\sin\theta'\sin\phi', R\cos\theta')$$

$$P_x = PR^2 \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \left(R \sin\theta' \cos\phi' \right) \cos\theta' \sin\theta'$$

$$\int_0^{2\pi} d\phi' \cos\phi' = 0$$

$$P_x = 0 = P_y$$

$$P_z = PR^2 \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \left(R \cos\theta' \right) \cos\theta' \sin\theta'$$

$$= 2\pi PR^3 \underbrace{\int_0^{\pi} \cos^2\theta' \sin\theta' \, d\theta'}_{2/3}$$

$$P_z = \frac{4}{3}\pi R^3 P = PV$$ ✓

Why did this field turn out so nice?

Answer: $\cos\theta = P_1(\cos\theta)$ the first Legendre polynomial.

So we can calculate the field of some simple distributions. Now, let's handle an arbitrary distribution

The potential of a dipole is

$$V_{dip}(\vec{r}) = \frac{\hat{r}'' \cdot \vec{p}}{4\pi\varepsilon_0 r''^2}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\hat{r}'' \cdot \vec{P}(\vec{r}')}{r''^2}\, dv'$$

Recall

$$\nabla'\left(\frac{1}{r''}\right) = \frac{\hat{r}''}{r''^2}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \vec{P}(\vec{r}') \cdot \nabla'\left(\frac{1}{r''}\right) dv'$$

Use vecter identity

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_V \nabla' \cdot \left(\frac{\vec{P}}{r''}\right) dv' - \int_V \frac{1}{r''}\left(\nabla' \cdot \vec{P}\right) dv'\right]$$

Divergence Thm

$$V = \frac{1}{4\pi\varepsilon_0} \oint_S \frac{1}{r''} \vec{P}\cdot d\vec{a}' - \frac{1}{4\pi\varepsilon_0} \int_V \frac{1}{r''} (\vec{\nabla}'\cdot\vec{P})\, dv'$$

Compare with point charge potential,

$$V = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}')\, dv'}{r''}$$

The first term looks like the potential of a surface charge, the second that of a volume charge.

Define Bound charge densities

$$\sigma_b = \vec{P}\cdot\hat{n} \qquad \rho_b = -\nabla\cdot\vec{P}$$

$$(d\vec{a} = da\,\hat{n})$$

then

$$V = \frac{1}{4\pi\epsilon_0}\oint_S \frac{\sigma_b \, da'}{r''} + \frac{1}{4\pi\epsilon_0}\int \frac{\rho_b \, dv'}{r''}$$

Note, for both the slab ($\sigma_b = P$) and the ball ($\sigma_b = P \cdot \hat{n} = P\cos\theta$) this is consistent with our models earlier.