Potential

Since $\nabla \cdot \vec{B} = 0$, the magnetic field is uniquely defined by the curl of some function $\vec{A}$.

<u>Vector Potential</u> ($\vec{A}$) - A function such that

$$\vec{B} = \nabla \times \vec{A}$$

<u>Use Faraday's Law</u>

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t} \nabla \times \vec{A}$$

$$= -\nabla \times \frac{\partial \vec{A}}{\partial t}$$

so $$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

which implies ( ) can be represented by the gradient of some function.

Scalar Potential (or just potential) - A function such that

$$-\nabla V = \vec{E} + \frac{\partial \vec{A}}{\partial t}$$

or $$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

Note, if we have V and $\vec{A}$ we automatically solve two of the Maxwell equations.

$$\nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) \equiv 0 \quad \text{always}$$

$$\nabla \times \vec{E} = \underbrace{-\nabla \times \nabla V}_{0} - \frac{\partial}{\partial t} \nabla \times \vec{A}$$

$$= -\frac{\partial \vec{B}}{\partial t}$$

With the other two we are not so lucky.

Gauss' Law

$$\nabla \cdot \vec{E} = \rho/\varepsilon_0$$

$$-\nabla^2 V - \frac{\partial}{\partial t} \nabla \cdot \vec{A} = \rho/\varepsilon_0$$

Ampere's Law

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \left[-\nabla \frac{\partial V}{\partial t} - \frac{\partial^2 \vec{A}}{\partial t^2} \right]$$

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

With some rearranging we get

$$\left(\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} \right) - \nabla \left(\nabla \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} \right)$$

$$= -\mu_0 \vec{J}$$

Yikes.

Fields are measurable, potentials are not. We already know V is undefined up to a constant. Let's see what latitude we have in choosing V and A.

Let $\vec{A}$, $\vec{A}'$ and V, V' be combinations of potentials that yield the same field.

$$\vec{A}' = \vec{A} + \vec{\alpha} \qquad \text{where} \qquad \vec{\alpha} = \vec{A}' - \vec{A}$$

To yield the same field, $\nabla \times \vec{\alpha} = 0$.

As before, this means $\vec{\alpha}$ can be represented by some scalar function, λ, s.t.

$$\vec{\alpha} = \nabla \lambda$$

The two functions must also give the same electric field. If $V' = V + B$

and $$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{E} = -\nabla V - \nabla B - \frac{\partial \vec{A}}{\partial t} - \frac{\partial \vec{\alpha}}{\partial t}$$

$$-\nabla B - \frac{\partial \vec{\alpha}}{\partial t} = 0$$

$$\nabla B + \frac{\partial \nabla \lambda}{\partial t} = 0$$

$$\nabla \left(B + \frac{\partial \lambda}{\partial t} \right) = 0$$

$$B = -\frac{\partial \lambda}{\partial t}$$

So the fields are unchanged if the potentials are changed by a scalar function λ by

$$\vec{A}' = \vec{A} + \nabla \lambda$$

$$V' = V - \frac{\partial \lambda}{\partial t}$$

This transformation is called a <u>gauge transformation.</u>

Coulomb Gauge - In magnetostatics, choose

$$\nabla\cdot\vec{A}=0$$

$$\nabla\cdot\vec{A}' = \nabla\cdot\vec{A} + \nabla^2\lambda$$

so $\nabla\cdot\vec{A} = -\nabla^2\lambda$ and let λ independent of time.

In the Coulomb Gauge,

Gauss $\quad \nabla^2 V = -\rho/\varepsilon_0$

Ampere $\quad \nabla^2\vec{A} - \mu_0\varepsilon_0 \dfrac{\partial^2\vec{A}}{\partial t^2} - \mu_0\varepsilon_0 \dfrac{\partial}{\partial t}\nabla V$

$$= -\mu_0\vec{J}$$

if fields are constant in time

$$\nabla^2\vec{A} = -\mu_0\vec{J}$$

Note, this gauge is not causal. Effects propagate instantly.

Lorentz Gauge $$\nabla \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} = 0$$

Gauss $$\nabla^2 V - \mu_0 \varepsilon_0 \frac{\partial^2 V}{\partial t} = -\rho/\varepsilon_0$$

Ampere $$\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

$\Rightarrow$ Wave equations $v = c$

Define d'Alombertion operator

$$\Box^2 \equiv \nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}$$

$$\Box^2 V = -\rho/\varepsilon_0$$

$$\Box^2 \vec{A} = -\mu_0 \vec{J}$$

Let's make sure the Lorentz gauge is a valid gauge choice.

$$V' = V - \frac{\partial \lambda}{\partial t}$$

$$\vec{A}' = \vec{A} + \nabla \lambda$$

$$\nabla \cdot \vec{A}' + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} = 0 = \nabla \cdot \vec{A} + \nabla^2 \lambda + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 \lambda}{\partial t^2}$$

Suppose $\nabla \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} \neq 0 = f(\vec{r}, t)$

Select λ s.t.

$$\nabla^2 \lambda - \mu_0 \varepsilon_0 \frac{\partial^2 \lambda}{\partial t^2} = -f(\vec{r}, t) = \Box^2 \lambda$$

which leaves $\nabla \cdot \vec{A}' + \mu_0 \varepsilon_0 \frac{\partial V'}{\partial t} = 0$

But $\Box^2 \lambda = -f$ is just a 3 dimensional wave equation for which we have a complete solution. So we can always find λ to satisfy Lorentz gauge.

Computing Potentials

The electric potential of a point charge is found

from $$-\nabla V = \frac{kq}{r^2}\hat{r}$$

$$-\frac{\partial V}{\partial r}\hat{r} = \frac{kq}{r^2}\hat{r}$$

Integrate, or just observe

$$V(r) = \frac{kq}{r} + C$$

Choose C so $V(\infty) = 0 \Rightarrow C = 0$

$$V(r) = \frac{kq}{r}$$

The potential is additive (from superposition) and continuous.

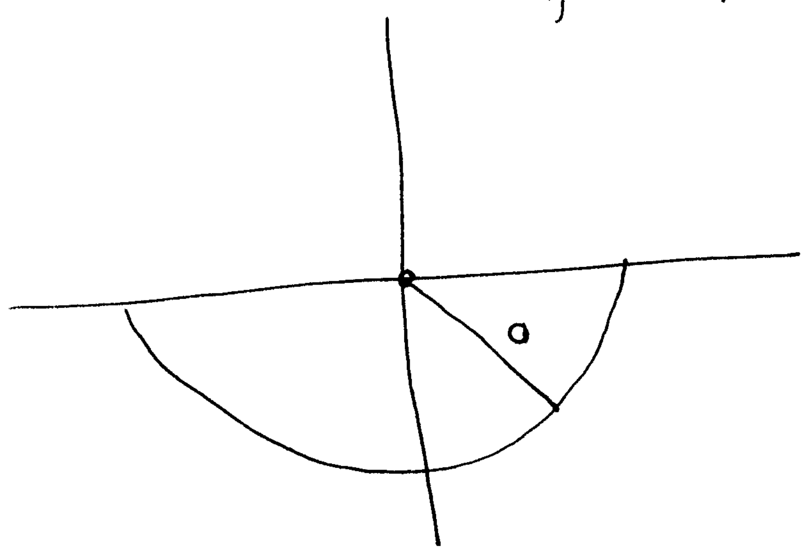
Therefore the potential of any distribution of charge is given by

$$V(\vec{r}) = \int \frac{k\rho(\vec{r}')\,d\tau'}{r''}$$ electrostatic

Which is in principle much easier to calculate.

Electric potential increases opposite the electric field like -OR- electric field points to lower potential.

Ex Compute potential of half circle at origin. Linear charge density λ.

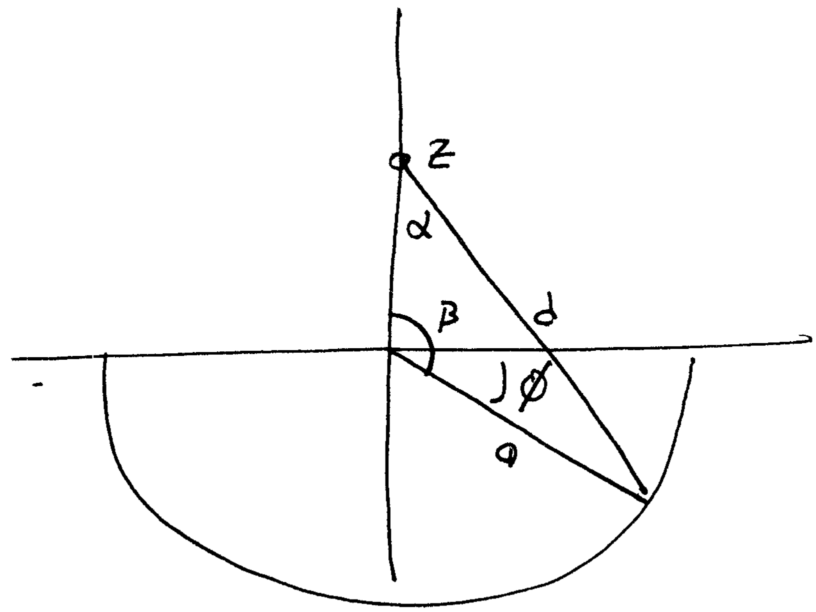


$$V(0) = k\int \frac{\lambda \, d\ell'}{a} \qquad r'' = a$$

$$= 2\pi k\lambda$$

So $\vec{E}(0) = -\nabla V = 0$? no. We need V as a function of z.

Ex



$$d^2 = a^2 + z^2 - 2az\cos\beta$$ Law of Cosines

$$\cos\beta = \cos\left(\phi + \pi/2\right) = -\sin\phi$$

$$V(z) = k\int_0^{\pi} \frac{d\phi\, a\lambda}{\sqrt{a^2+z^2+2az\sin\phi}}$$

~~Let $\theta = \phi/2$ $\cos\phi = 1 - 2\sin^2\theta$~~

Which is going to turn into an elliptical integral.

Calculating Vector Potential (Magnetostatic)

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \hat{r}''}{r''^2} dv' \qquad (\text{static})$$

Consider

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{J(\vec{r}\,')\, dv'}{r''}$$

$$\nabla \times \vec{A} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\frac{J(\vec{r}\,')}{r''} \right) dv'$$

The curl can come in the integral because the integration is over $\vec{r}\,'$, but the curl is with respect to $\vec{r}$.

Vector identity $\quad \nabla \times (f\vec{A}) = f(\nabla \times \vec{A}) - \vec{A} \times \nabla f$

$$\nabla \times \vec{A} = \frac{\mu_0}{4\pi} \int \left(\frac{1}{r''} \nabla \times \vec{J} - \vec{J} \times \nabla \left(\frac{1}{r''} \right) \right) dv'$$

$\nabla \times \vec{J}(\vec{r}\,') = 0 \quad$ different coordinates

$$\nabla \left(\frac{1}{r''} \right) = -\frac{\hat{r}''}{r''^2}$$

$$\nabla \times \vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \hat{r}''}{r''^2} dv' = \vec{B} \checkmark$$

~~or~~

We can re-write $\vec{A}$ for our other current sources.

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{I\, d\vec{l}'}{r''} \qquad \vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{K}\, da'}{r''}$$

A useful property of $\vec{A}$

$$\int_S (\nabla \times \vec{A}) \cdot d\vec{a} = \int_S \vec{B} \cdot d\vec{a} \equiv \Phi_m$$

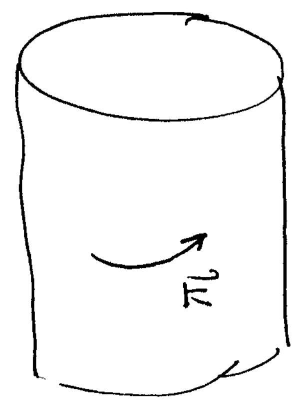
$$= \oint_C \vec{A} \cdot d\vec{l} \qquad \text{Stokes}$$

The magnetic flux is

$$\Phi_m = \oint_C \vec{A} \cdot d\vec{l}$$

Ex A infinite solenoid has current $K = N'I$.

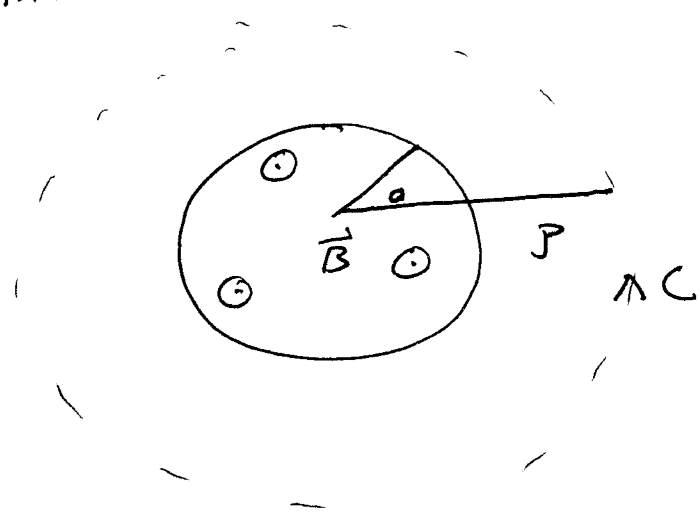
Compute $\vec{A}$



$B = N' I \mu_0$

$= \mu_0 K$

Outside the solenoid



C chosen so

$\Phi > 0$.

By symmetry, $\vec{A}$ circular.

$$\Phi_m = \pi a^2 B = \oint_C \vec{A} \cdot d\vec{l} = 2\pi\rho A$$

$$A(\rho) = \frac{\pi a^2 B}{2\pi\rho}$$

$$\vec{A} = \frac{a^2 B}{2\rho}\hat{\phi}$$

Inside the solenoid,

$$\Phi_m = \pi\rho^2 B$$

$$\vec{A} = \frac{\pi\rho^2 B}{2\pi\rho}\,\hat{\phi}$$

$$= \frac{\rho B}{2}\,\hat{\phi}$$

<u>Check</u> Outside

$$\nabla\times\vec{B} = \frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial\rho}\right)\hat{z}$$

$$= 0$$

<u>Inside</u> $\nabla\times\vec{B} = \frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial\rho}\right)\hat{z}$

$$= \frac{1}{\rho}\,\frac{\partial}{\partial\rho}\left(\frac{\rho^2 B}{2}\right)\hat{z} = \vec{B}\hat{z} \checkmark$$