Spherical Systems

Now consider systems with spherical symmetry.

The Laplacian is

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right)$$

$$+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

As before, let's start with systems without ϕ dependence. Again propose a separated solution

$$V(r,\theta) = R(r)\,\Phi(\theta)$$

Solutions $\quad 1, \; r, \; \theta, \; \ln(\tan \theta/2)$

$$r^n P_n(\cos\theta) \qquad r^{-(n+1)} P_n(\cos\theta)$$

$P_n(x)$ Legendre Polynomials

$$P_0(x) = 1 \qquad P_1(x) = x$$

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

Orthogonality Relation

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \frac{2}{2n+1}\,\delta_{mn}$$

or

$$\int_{-1}^{1} P_n(\cos\theta)\, P_m(\cos\theta)\, d(\cos\theta) = \frac{2}{2n+1}\,\delta_{mn}$$

General Spherical Solution Laplace's Eqn (ϕ not constant)

$$V(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{lm} r^{l} + B_{lm} r^{-(l+1)} \right] Y_l^m(\phi,\theta)$$

Spherical Harmonics

$$Y_l^m(\phi,\theta) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}}\; P_l^m(\cos\theta)\, e^{im\phi}$$

Associated Legendre Polynomials.

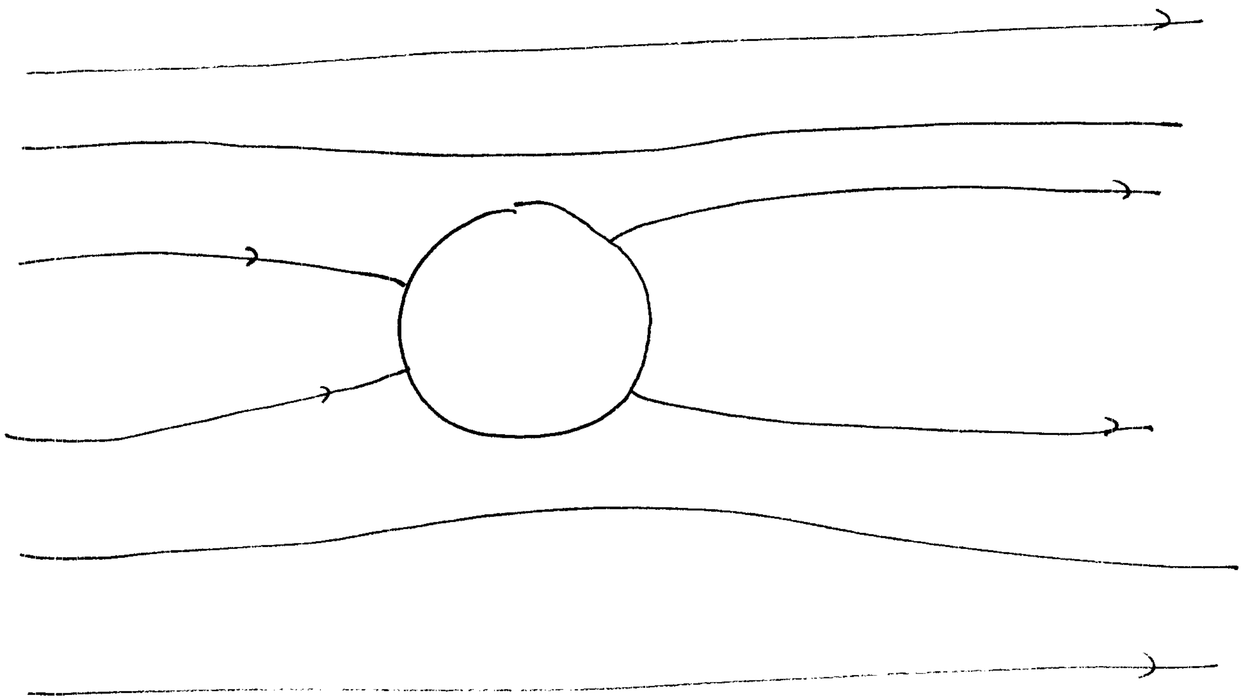
$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \; Y^*_{l'm'} Y_{lm} = \delta_{l'l}\, \delta_{m'm}$$

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\phi} \qquad Y_1^{-1} = \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\phi}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta$$

Ex (From Good) - Conducting sphere in uniform field $\vec{E} = E_0 \hat{x}$



<u>Boundary Condition</u> Far from sphere field is

$$\vec{E} = E_0 \hat{x} \Rightarrow V = -E_0 x$$

$$V = -E_0 r\cos\theta$$ (Let θ measured from x axis)

Inside sphere $\vec{E} = 0$ - At surface $\vec{E} \cdot \hat{t} = 0$

$\int \vec{E} \cdot \hat{n}\, dA = 0$

Outside sphere

$$V(r,\theta) = \sum_n A_n r^{-(n+1)} P_n(\cos\theta) + B_n r^n P_n(\cos\theta)$$

Apply Boundary Conditions

At large r, $\quad V(r,\theta) \rightarrow -E_0 r\cos\theta$

$$B_n = 0 \quad \text{if} \quad n \neq 1$$

$$B_1 = -E_0$$

since $\quad P_1(\cos\theta) = \cos\theta$

Now, the surface of the conductor must all be at the same potential. Let this potential be 0.

$$V(a,\theta) = 0 = \sum A_n a^{-(n+1)} P_n(\cos\theta) + \left(-E_0 a P_1(\cos\theta)\right)$$

For $V=0$, the A_n term must vary with the same angular dependence. $\quad A_n \neq 0 \quad n=1, \quad A_n = 0 \quad n \neq 1$

$$V(a,\theta) = 0 = A_1 a^{-2} P_1(\cos\theta) - E_0 a P_1(\cos\theta)$$

$$A_1 = a^3 E_0$$

The potential is then

$$V(r,\theta) = -E_0 r\cos\theta + E_0 \frac{a^3\cos\theta}{r^2}$$

continued next page.

<u>Ex</u> Dielectric Sphere in Uniform Field (radius a)

<u>Boundary Conditions</u>

(1) As $r \to \infty$, $V(r,\theta) = -E_0 r\cos\theta = -E_0 r P_1(\cos\theta)$

(2) $V_i(a,\theta) = V_0(a,\theta) \Rightarrow$ Potential continuous.

(3) There is no free charge at the dielectrics surface so D_{normal} ~~const~~ continuous.

$$D_{normal} = \hat{r}\cdot\vec{D} = \hat{r}\cdot(\varepsilon_0\varepsilon_r\vec{E})$$

$$= -\varepsilon_0\varepsilon_r \frac{\partial V}{\partial r}\bigg|_a$$

Continued in two pages.

Finishing Sphere Problem

Check (1) is the field normal to the surface

$\Rightarrow$ should be since surface equipotential

(2) is the total charge on the conductor zero.

(1) $$\vec{E} = -\nabla V = -\left(\frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}\right)$$

$$= \left(E_0 \cos\theta + \frac{2E_0 a^3 \cos\theta}{r^3}\right)\hat{r}$$

$$-\left(\frac{E_0 r \sin\theta}{r} - \frac{E_0 a^3 \sin\theta}{r^3}\right)\hat{\theta}$$

At the surface,

$$\vec{E}(a) = \left(E_0 \cos\theta + 2E_0 \cos\theta\right)\hat{r}$$

$$-\left(E_0 \sin\theta - E_0 \sin\theta\right)\hat{\theta}$$

$$= 3E_0 \cos\theta\,\hat{r}$$ which is $\perp$ surface

(2) Total charge on sphere

$$\sigma = \varepsilon_0 (\vec{E} \cdot \hat{r}) = 3 E_0 \cos\theta \, \varepsilon_0$$

$$Q = \int \sigma \, da$$

$$= \int_0^{2\pi} \int_0^{\pi} (a \sin\theta \, d\phi)(a \, d\theta) \, \sigma$$

$$= 3 E_0 a^2 \varepsilon_0 \int_0^{2\pi} \int_0^{\pi} \sin\theta \cos\theta \, d\theta \, d\phi$$

$$= 6\pi E_0 a^2 \varepsilon_0 \int_0^{\pi} \sin\theta \cos\theta \, d\theta$$

$$= 3\pi E_0 a^2 \varepsilon_0 \int_0^{\pi} \sin 2\theta \, d\theta \qquad \sin 2\theta = 2 \sin\theta \cos\theta$$

$$= 0$$

Return to Dielectric Sphere

Solution Outside

$$V_o(r,\theta) = \sum_n A_n r^{-(n+1)} P_1(\cos\theta) - E_0 r P_1(\cos\theta)$$

$$V_i(r,\theta) = \sum C_n r^n P_n(\cos\theta)$$

Continuous

$$V_o(a,\theta) = V_i(a,\theta)$$

$$\sum C_n a^n P_n(\cos\theta) = -E_0 a P_1(\cos\theta) + \sum A_n a^{-(n+1)} P_n(\cos\theta)$$

$\Longrightarrow$ Match like powers

$$C_n a^n = A_n a^{-(n+1)} \qquad n \neq 1$$

$$C_1 a = -E_0 a + A_1 a^{-2}$$

Derivative (Dnormal Continuous)

$$\varepsilon_r \frac{\partial V_i}{\partial r}\bigg|_a = \frac{\partial V_o}{\partial r}\bigg|_a$$

$$\varepsilon_r \sum C_n n r^{n-1} P_n(\cos\theta)\bigg|_a = -E_o P_1(\cos\theta) + \sum A_n(-(n+1))\, r^{-(n+2)} P_n(\cos\theta)\bigg|_a$$

Once again match like powers of $\cos\theta$

$$\varepsilon_r C_1 = -E_o - A_1 2 a^{-3} \qquad n=1$$

and

$$\varepsilon_r C_n n a^{n-1} = -(n+1) A_n a^{-(n+2)} \qquad n \neq 1$$

Solve $n \neq 1$

$$\varepsilon_r n C_n a^n = -(n+1) A_n a^{-(n+1)}$$

$$\varepsilon_r n A_n a^{-(n+1)} = -(n+1) A_n a^{-(n+1)}$$

$\varepsilon_r n = -(n+1)$ which does not have a general integer solution, $A_n = C_n = 0$

Now solve $n=1$

$$\left.\begin{aligned} C_1 &= -E_0 + \frac{A_1}{a^3} \\ \varepsilon_r C_1 &= -E_0 - \frac{2A_1}{a^3} \end{aligned}\right] \text{Solve}$$

$$-\varepsilon_r E_0 + \frac{\varepsilon_r A_1}{a^3} = -E_0 - \frac{2A_1}{a^3}$$

$$E_0(1-\varepsilon_r) = -\frac{A_1}{a^3}(2+\varepsilon_r)$$

$$A_1 = a^3 E_0 \frac{(\varepsilon_r - 1)}{2+\varepsilon_r} \qquad C_1 = -E_0 + E_0\frac{(\varepsilon_r-1)}{2+\varepsilon_r}$$

$$= \frac{E_0}{2+\varepsilon_r}\left(-2-\varepsilon_r+\varepsilon_r-1\right)$$

$$= \frac{-3E_0}{2+\varepsilon_r}$$

Total Solution

$$V_i = \frac{-3E_0}{2+\varepsilon_r} r\cos\theta$$

$$V_o = -E_0 r\cos\theta + \frac{a^3}{r^2}E_0\cos\theta\frac{(\varepsilon_r-1)}{2+\varepsilon_r}$$

Spherical Coordinates - Trivial Solutions

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial V}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right)$$

$$+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

~~R~~ r-dependance alone

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial V}{\partial r} = 0$$

$$r^2 \frac{\partial V}{\partial r} = A$$

$$\frac{\partial V}{\partial r} = \frac{A}{r^2}$$

$$V = \frac{1}{r}$$

ϕ Dependance alone

$$\frac{\partial^2 V}{\partial \phi^2} = 0 \implies V = \phi$$

θ Dependence Alone

$$\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial V}{\partial\theta} = 0$$

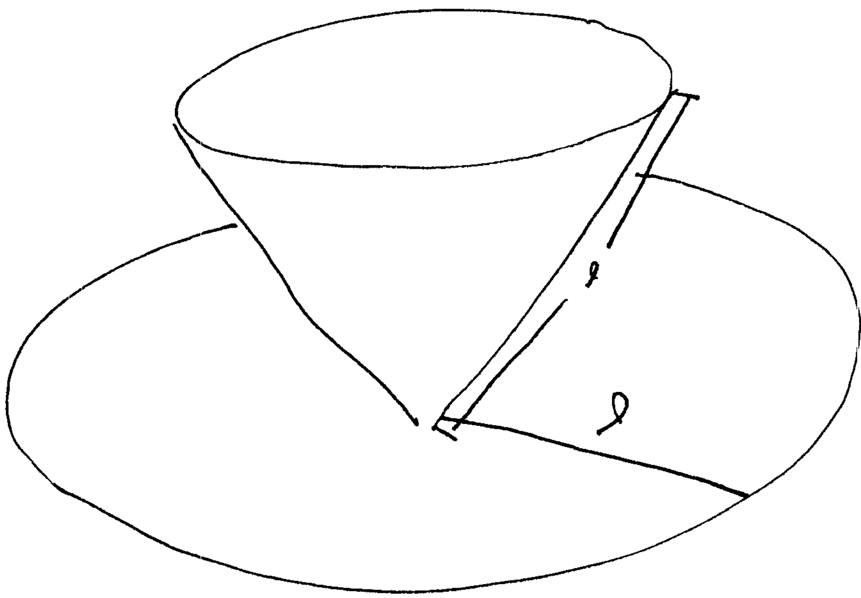
$$\frac{\partial V}{\partial\theta} = \frac{A}{\sin\theta}$$

$$V = A \ln\left(\csc(\theta) - \cot(\theta)\right) + B$$

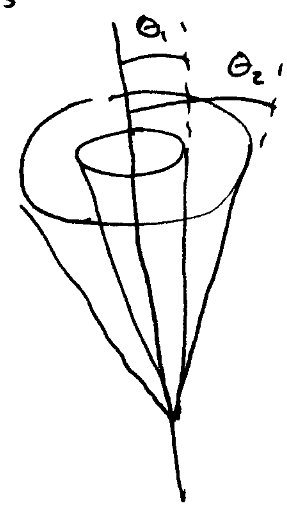
$$= A \ln\left(\frac{\sin(\theta)}{\cos(\theta)+1}\right) + B$$

$$= A \ln\left(\tan \theta/2\right) + B$$

$\underline{\text{Ex}}$ Compute field / capacitance of $45°$ cone, height $\frac{l}{\sqrt{2}}$ above circular disk of radius l. No fringing (big assumption). Let disk be held at V_0, cone at 0.



Same type of system as



Try solution with no r, ϕ dependance

$$V(\theta) = A \ln\left(\tan \theta/2\right) + B$$

Boundary conditions

$$V(45^\circ) = 0$$

$$V(90^\circ) = V_0$$

$\Rightarrow$ We will automatically satisfy $\vec{E} \parallel \hat{n}$ because the surfaces are equipotentials.

$$V(45) = A \ln\left(\tan \frac{\pi}{8}\right) + B = 0$$

$$V(90) = A \ln\left(\tan \frac{\pi}{4}\right) + B$$

$$= A \ln(1) + B = B = V_0$$

$$A = \frac{-V_0}{\ln\left(\tan \pi/8\right)}$$

$$V(\theta) = -V_o \left(\frac{\ln(\tan \theta/2)}{\ln(\tan \pi/8)} - 1 \right)$$

The electric field is

$$\vec{E} = -\nabla V = -\frac{1}{r}\frac{\partial V}{\partial \theta}\,\hat{\theta}$$

$$= \frac{-V_o}{\ln(\tan \pi/8)} \frac{\cancel{\sin\theta}}{r\sin\theta}\,\hat{\theta} = \frac{-V_o}{\gamma\, r\sin\theta}\,\hat{\theta}$$

$$\gamma \equiv \ln(\tan \pi/8)$$

The charge density on the flat disk is

$$\sigma = -\frac{V_o}{\gamma r}\varepsilon_o \quad \text{(using Gaussian Pillbox)}$$

The total charge on the disk is

$$Q = \int_0^{2\pi}\int_0^{\ell} r\, d\theta\, dr\, \sigma$$

$$= -\frac{V_o}{\gamma}\int_0^{2\pi}\int_0^{\ell} d\theta\, dr = \frac{-V_o 2\pi \ell}{\gamma}$$

Capacitance

$$C = \left|\frac{Q}{V_0}\right| = \frac{2\pi\ell}{\ln\left(\tan\frac{\pi}{8}\right)}$$