PHYS 3414 - Electricity and Magnetism- Homework Set 5

Chapter 5 - Maxwell's Equations and Review

This set will not be collected, but should be completed before the test at 12:30pm Monday February 15, 2008.

Good's Problems

5.1

5.2

5.3

5.4

5.5

5.6

5.8

5.12

Exam Review Questions

Most of the above are excellent review questions. The following questions would be considered good test questions. These would be fairly difficult test questions.

1 Consider a spherical volume charge with uniform volume charge density Γ and radius a. The volume has a cavity centered at $a/4\hat{x}$ with radius $a/2$. Compute the electric force on a charge q placed at the center of the cavity.

2 An infinite slab of current occupying the region $-a < z < a$ has current density $\vec{J} = J_0\hat{x}$. Compute the magnetic field everywhere. Compute the magnetic force on a charge q moving with velocity $\vec{v} = v_0\hat{z}$ at the point $\vec{r} = (0, 0, a)$.

3 Consider an electric and magnetic field confined to the cylindrical region $\rho < a$.

$$\vec{E} = \gamma_0 \rho^2 \hat{\rho} \qquad \vec{B} = \gamma_1 \rho \hat{z}$$

Is this combination a possible electromagnetic field? If yes find the charge density and current. In both cases support your choice.

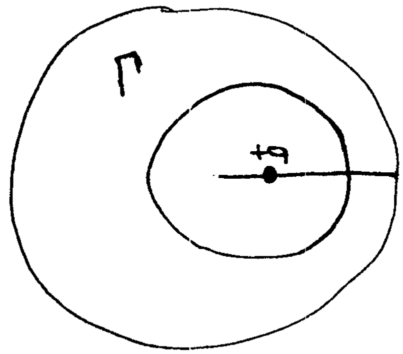
4 A non-uniformly charged disk with surface charge density $\sigma = \lambda\rho$ and radius a is parallel to an infinite plane with charge density σ_p. Calculate the total force exerted by the plane on the disk.

5 Compute the magnetic field along the axis of a disk with non-uniform charge density $\sigma = \lambda\rho$ rotating with an angular velocity ω about its axis.

6 Compute the field along the axis of a non-uniform disk of charge with radius a and charge density $\sigma = \lambda\rho$.

A1

Ex Consider a volume charge with uniform charge density Γ for $r<a$. The volume charge has a cavity at $\vec{r} = a/4\,\hat{x}$ with radius $a/2$. Compute the force on a positive charge at the center of the cavity.



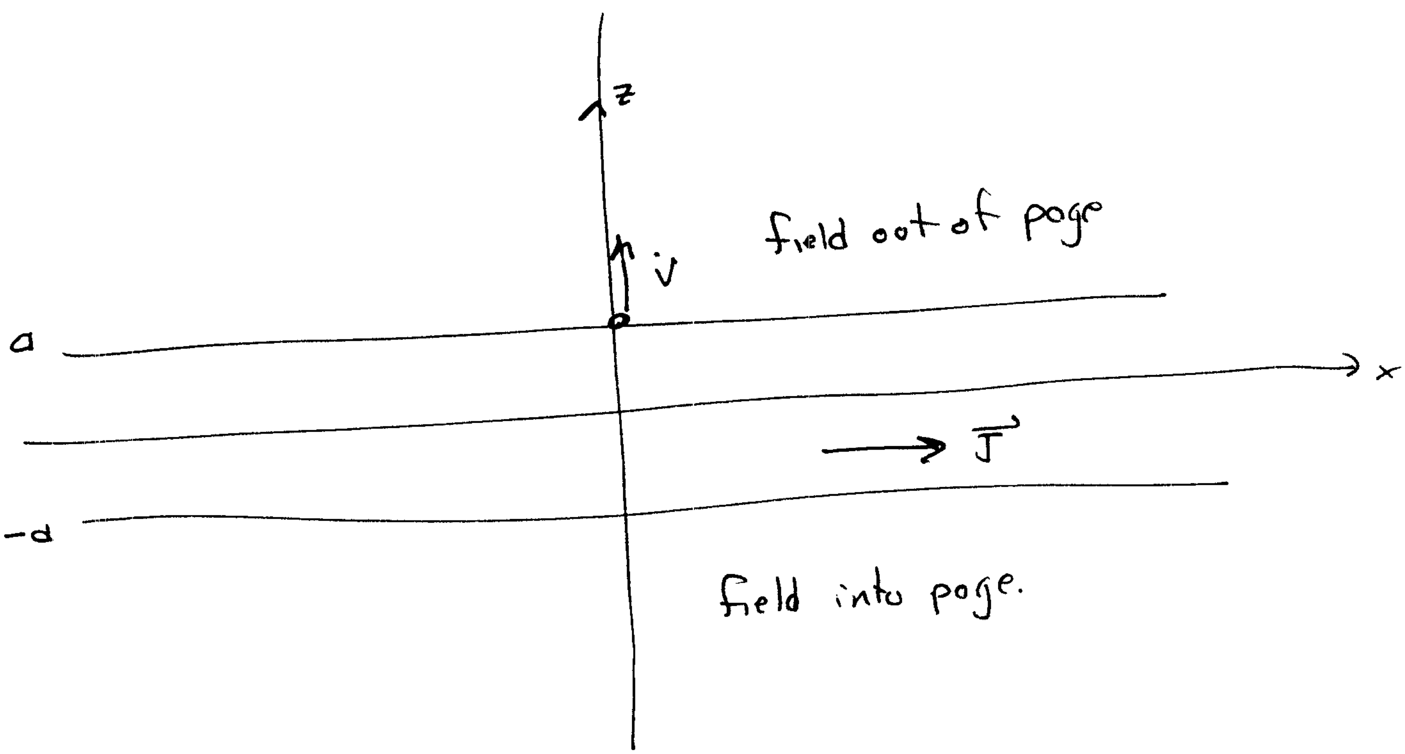
The electric field at the center of the cavity is the sum of the field of the solid volume charge without the cavity + The field of a solid volume charge of charge density $-\Gamma$ centered at $a/4\,\hat{x}$.

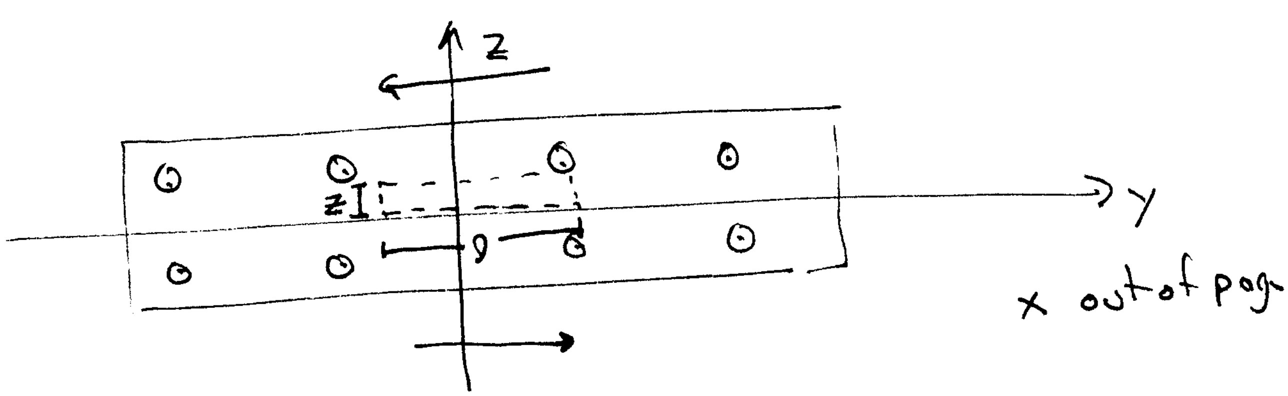
The field of the negative charge is zero at its center, so the force is just $\vec{F} = q\vec{E}_{solid} = \dfrac{q r \Gamma}{3\epsilon_0}\hat{x}$

$$= \frac{q\Gamma(a/4)}{3\epsilon_0}\hat{x}$$

Ex (A2) A slab of current $-a < z < 0$ has current density $\vec{J} = J_0 \hat{x}$. Compute the magnetic field and the magnetic force on a charge q travelling with velocity $\vec{v} = v_0 \hat{z}$ at $\vec{r} = (0,0,a)$.

Sln





Field points in $-y$ direction $z>0$, $+y$ direction $z<0$.

$$\vec{B} = B(y)\,\hat{y} \qquad B(0)=0.$$

Region ~~$z<a$~~ $z>0$

Use Amperian path of length l with one side along axis. Current encircled $I_{enc} = l z J_0$

Ampere's Law $\oint \vec{B}\cdot d\vec{l} = -B(z)\,l = \mu_0 I_{enc} = l z J_0 \mu_0$

$$\vec{B}(z) = -J_0 \mu_0 z\, \hat{y}$$

Region $z>-a$, $z<0$

$$\vec{B}(z) = -J_0 \mu_0 z \hat{y} \quad \text{by symmetry.}$$

Region $z>a$

$$I_{enc} = l a J_0$$

$$\oint \vec{B}\cdot d\vec{l} = -B(z)\,l = l a J_0 \mu_0$$

$$\vec{B}(z) = -a J_0 \mu_0 \hat{y}$$

Region $z<-a$

$$\vec{B}(z) = a J_0 \mu_0 \hat{y}$$

Force on moving charge,

$$\vec{F} = q\,\vec{v} \times \vec{B} = q v_0\, \hat{z} \times \left(-a J_0 \mu_0 \hat{y} \right)$$

$$= q v_0 a J_0 \mu_0\, \hat{x}$$

Ex (A3) Consider an electric and magnetic field confined to the ~~sp~~ cylindrical region $\rho < a$.

$$\vec{E} = \gamma_0 \rho^2 \hat{\rho} \qquad \vec{B} = \gamma_1 \rho \hat{z}$$

Are these possible EM fields. If yes describe the currents and charge densities. If no, tell why

Sln Check Maxwell

Gauss
$$\nabla \cdot \vec{E} = \frac{1}{\rho}\left(\frac{\partial \rho A_\rho}{\partial \rho}\right)$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\gamma \rho^3 = \frac{3\gamma\rho^2}{\rho} = 3\gamma\rho$$

$$= \Gamma / \varepsilon_0 \quad \text{where } \Gamma \text{ charge density}$$

$$\Gamma = 3\varepsilon_0 \gamma \rho$$

No Monopoles $\nabla \cdot \vec{B} = 0$ ✓

<u>Faraday</u>

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \times \vec{E} = 0 \quad \checkmark$$

<u>Ampere</u>

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

$$= \mu_0 \vec{J}$$

$$\nabla \times \vec{B} = -\frac{\partial A_z}{\partial \rho} \hat{\phi} = -\gamma_1 \hat{\phi} = \mu_0 \vec{J}$$

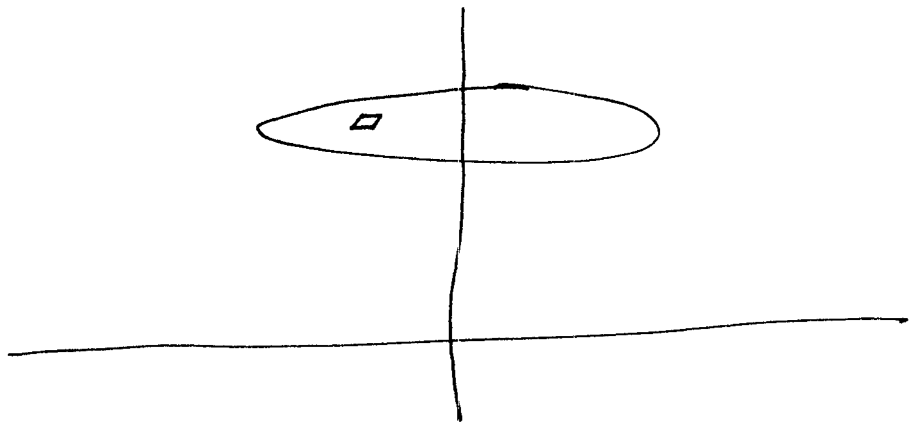
$$\vec{J} = -\frac{\gamma_1}{\mu_0} \hat{\phi}$$

So these can be EM fields.

(A4)

Ex A non-uniformly charged disk with charge density $\sigma = \lambda \rho$ is parallel to an infinite plane with charge density σ_p.

Compute the force exerted on the disk.



The field at some point on the disk is the sum of the field from the plane and the field from other elements of the disk. The ~~forces~~ internal forces of the disk are ignored.

The field of the plane is $\vec{E}_p = \dfrac{\sigma_p}{2\epsilon_0}\hat{z}$

The total force is $\vec{F} = \int dq\, \vec{E}_p$ $\qquad dq = \sigma\, da$

$$= \sigma \rho\, d\rho\, d\phi$$

$$= \int_0^{2\pi}\int_0^{a} \sigma \rho\, d\rho\, d\phi\, \vec{E}_p$$

$$\vec{F} = \int_0^{2\pi}\int_0^a \lambda \rho\, \rho\, d\rho\, d\phi\, \vec{E}_p$$

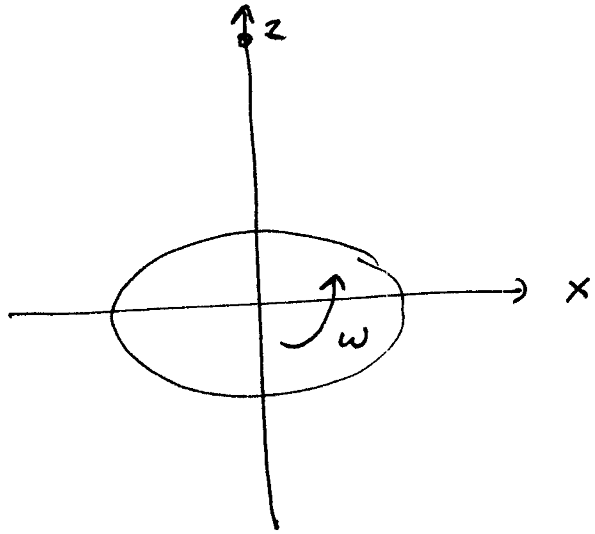
$$= \lambda \vec{E}_p \int_0^{2\pi}\int_0^a \rho^2 d\rho\, d\phi$$

$$= \frac{\lambda a^3}{3} 2\pi \vec{E}_p$$

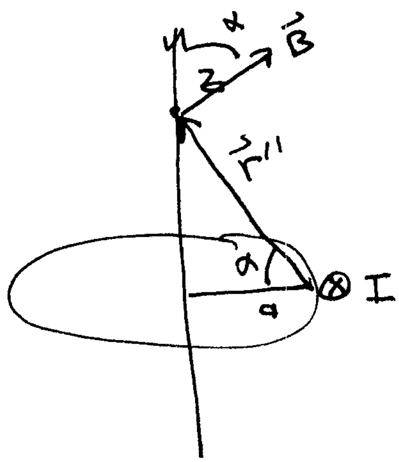
$$= \frac{2\pi\lambda a^3}{3} \frac{\sigma}{2\epsilon_0} \hat{z} = \frac{\pi\lambda a^3}{3\epsilon_0} \hat{z}$$

(A5)

<u>Ex</u> Compute the magnetic field of a non-uniform disk with charge density $\sigma = \lambda r$ rotating at angular velocity ω.



Magnetic field of current loop



$$\cos\alpha = \frac{a}{\sqrt{z^2+a^2}}$$

Field is in z-direction. All field magnitudes are the same

$$\vec{B} = \frac{\mu_0 I}{4\pi}\oint \frac{d\vec{l} \times \hat{r}''}{r''^2}$$

$$B_z = \frac{\mu_0 I \, 2\pi a}{4\pi (z^2+a^2)} \cos\alpha$$

$$B_z = \frac{\mu_0 I a^2}{2 (z^2+a^2)^{3/2}}$$

$$\vec{B} = \frac{\mu_0 I a^2}{2(z^2+a^2)^{3/2}} \hat{z}$$

For rotating disk, $I = \sigma \omega \rho' d\rho'$

since the velocity is $\omega \rho'$.

$$\vec{B} = \int_0^a \vec{B}_{ring}(\rho')\, d\rho'$$

$$= \frac{\mu_0 \cancel{I}}{2} \hat{z} \int_0^a \frac{(\sigma\omega\rho' d\rho')(\rho')^2}{(z^2+\rho'^2)^{3/2}}$$

$$= \frac{\mu_0 \sigma \omega \hat{z}}{2} \int_0^a \frac{\rho'^3 d\rho'}{(z^2+\rho'^2)^{3/2}}$$

What follows is correct if $\sigma =$ constant.

I will work the assigned problem after.

From integral table

$$\vec{B} = \frac{\mu_0 \sigma \omega}{2} \hat{z} \left[\sqrt{x^2+z^2} + \frac{z^2}{\sqrt{x^2+z^2}} \right]_0^a$$

$$= \frac{\mu_0 \sigma \omega}{2} \hat{z} \left[\sqrt{a^2+z^2} - |z| + \frac{z^2}{\sqrt{a^2+z^2}} - |z| \right]$$

See next page for $\sigma = \lambda p$ solution

Now, let $\sigma = \lambda \rho'$

$$\vec{B} = \frac{\mu_0}{2}\hat{z}\int_0^a \frac{(\lambda\rho'\omega\rho' d\rho')(\rho')^2}{(z^2+\rho'^2)^{3/2}}$$

$$= \frac{\mu_0\lambda\omega}{2}\hat{z}\int_0^a \frac{\rho'^4 d\rho'}{(z^2+\rho'^2)^{3/2}}$$

$$\vec{B} = \frac{\mu_0\lambda\omega}{2}\hat{z}\left(\frac{1}{2}\frac{\rho'^3}{\sqrt{z^2+x^2}} + \frac{3}{2}\frac{z^2\rho'}{\sqrt{z^2+\rho'^2}} - \frac{3}{2}\ln\left(x+\sqrt{z^2+x^2}\right)z^2\right)\Bigg|_0^a$$

$$= \frac{\mu_0\lambda\omega}{2}\hat{z}\left(\frac{1}{2}\frac{a^3}{\sqrt{a^2+z^2}} + \frac{3}{2}\frac{z^2 a}{\sqrt{z^2+a^2}} - \frac{3z^2}{2}\left(\ln\left(\frac{a+\sqrt{z^2+a^2}}{|z|}\right)\right)\right)$$

Maple Integral

A bit messy for a test, but you should be able to set up the integral

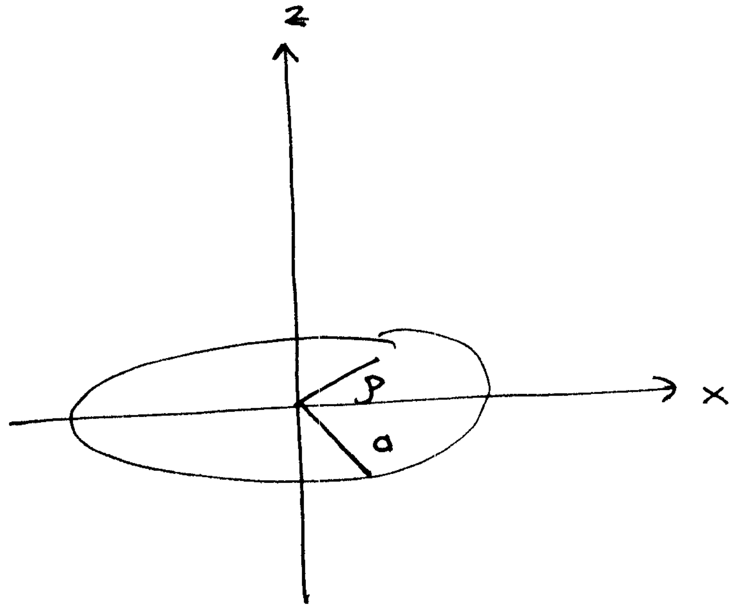
$$> \ int\left(\frac{x^4}{\left(z^2+x^2\right)^{\left(\frac{3}{2}\right)}},\ x\right);$$

$$\frac{1}{2}\,\frac{x^3}{\sqrt{z^2+x^2}}+\frac{3}{2}\,\frac{z^2\,x}{\sqrt{z^2+x^2}}-\frac{3}{2}\,z^2\ln\left(x+\sqrt{z^2+x^2}\right) \qquad (1)$$

$>$

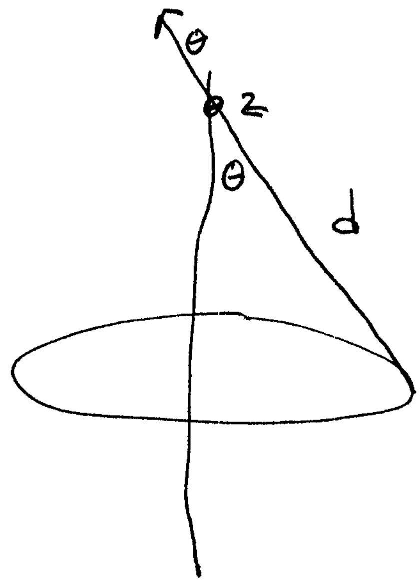
A6

Ex Compute the field of a non-uniform disk of charge with charge density $\sigma = \lambda \rho$



Field of a Ring

$$E_z = \frac{Q}{4\pi\varepsilon_0 d^2}\cos\theta$$



$$\cos\theta = \frac{z}{d}$$

$$E_z = \frac{Q\,z}{4\pi\varepsilon_0 (z^2+a^2)^{3/2}} = \frac{2\pi a\lambda z}{4\pi\varepsilon_0 (z^2+a^2)^{3/2}}$$

$$= \frac{a\lambda z}{2\varepsilon_0 (z^2+a^2)^{3/2}}$$

$$\vec{E} = \frac{a\lambda z}{2\varepsilon_0 (z^2+a^2)^{3/2}}\,\hat{z}$$

The field of a disk is

$$E_z = \int_0^a \frac{\sigma(\rho')\, d\rho'\, \rho' z}{2\varepsilon_0 (z^2+\rho'^2)^{3/2}}$$

$$= \frac{z}{2\varepsilon_0}\int_0^a \frac{\lambda \rho'^2\, d\rho'}{(z^2+\rho'^2)^{3/2}}$$

$$= \frac{z\lambda}{2\varepsilon_0}\left[\frac{-x}{\sqrt{x^2+z^2}} + \ln\left(x+\sqrt{x^2+z^2}\right)\right]_0^a$$

$$= \frac{z\lambda}{2\varepsilon_0}\left[\frac{-a}{\sqrt{a^2+z^2}} + \ln\left(\frac{a+\sqrt{a^2+z^2}}{\sqrt{z^2}}\right)\right]$$

(A7)

Ex Volume charged density oscillates as

$$\Gamma(r) = \Gamma_0 (\sin\omega t) r$$

in the region $r<a$. Compute $\vec{E}$, $\vec{J}$

Use Gauss Law $r<a$

$$Q_{enc} = \int_0^r (\Gamma_0 r \sin\omega t) 4\pi r^2 dr$$

$$= 4\pi \Gamma_0 \sin\omega t \int_0^r r^3 dr$$

$$= 4\pi \Gamma_0 \sin\omega t \frac{r^4}{4} = \pi \Gamma_0 \sin\omega t \, r^4$$

$$\Phi = 4\pi r^2 E(r) = \frac{Q_{enc}}{\varepsilon_0} \quad \text{Gauss}$$

$$\vec{E}(r) = \frac{4\pi \Gamma_0 \sin\omega t \, r^4}{16\pi \varepsilon_0 r^2} = \frac{\Gamma_0 r^2 \sin\omega t}{4\varepsilon_0} \hat{r}$$

Continuity Eqn

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\int_s \vec{J} \cdot d\vec{a} = -\frac{\partial}{\partial t} \int \rho \, dv$$

Current must be spherical $\qquad \vec{J} = J_0(r)\,\hat{r}$

$$\int \vec{J}\cdot d\vec{a} = 4\pi r^2 J_0(r) = -\frac{\partial}{\partial t} Q_{enc}$$

$$= -\omega \pi \Gamma_0 \cos\omega t \, r^4$$

$$J_0(r) = -\frac{\omega \Gamma_0 r^2 \cos\omega t}{4}$$

$$\frac{\partial \vec{E}}{\partial t} = +\frac{\omega \Gamma_0 r^2 \cos\omega t}{4\epsilon_0}\,\hat{r}$$

Ampere's Law

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$= \mu_0 \left(-\frac{\omega \Gamma_0 r^2 \cos\omega t}{4} \right) + \mu_0 \epsilon_0 \left(\frac{\omega \Gamma_0 r^2 \cos\omega t}{4\epsilon_0} \right)$$

$$= 0$$

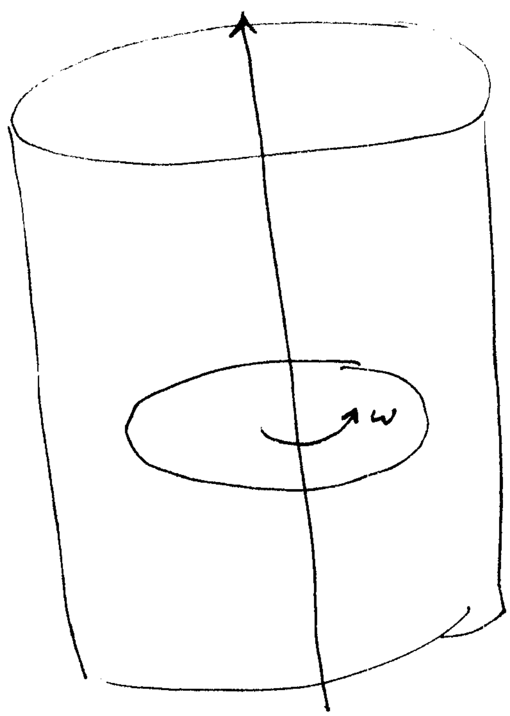
Faraday's Law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = 0 \quad \Longrightarrow \quad \vec{B} = 0$$

(A8)

Ex A uniformly charged disk is placed in a solenoid with N' turns per unit length carrying current I.

The disk has charge density σ and spins with angular velocity ω. Compute the force per unit area as a function of ρ. The axis of rotation is $\parallel$ to the axis of the solenoid.



$$\vec{B} = \mu_0 N' I_{sole}\,\hat{z} \equiv B_0\,\hat{z}$$

$$d\vec{l} = \rho\,\hat{\phi}\,d\phi$$

$$d\vec{F}_m = I_{disk}\,d\vec{l} \times \vec{B}$$

$$d\vec{l} \times \vec{B} = \rho\,d\phi\,B_0\,\hat{\rho}$$

$$= (\sigma\omega\rho\,d\rho)\,\rho\,d\phi\,B_0\,\hat{\rho}$$

$$I = \sigma v\,d\rho = \sigma\omega\rho\,d\rho$$

$$da = d\rho\,\rho\,d\phi$$

$$d\vec{F}_m = \sigma\omega\rho\,B_0\,\hat{\rho}\,da$$

$$\text{Force / area} = \sigma\omega\rho\,B_0\hat{\rho} = \sigma\omega\rho\,\mu_0 N' I_{sole}\,\hat{\rho}$$