PHYS 3414 - Electricity and Magnetism- Test 1 - Part 1

All problems are worth 25 points. The majority of points on each problem will be awarded for doing the physics correctly; if you have correctly done the physics, but cannot carry out the mathematics, you will still receive most of the points. Final tournament score : Woodland 25 - Bentonville Black 17. Woodland finishes season as undefeated regional champions. I wrote the test after the tournament.

1 A non-uniform semi-infinite cylindrical volume charge with volume charge density $\Gamma = \gamma\rho^2$ occupies the region $\rho < a$. Compute the electric field inside ($\rho < a$) and the field outside ($\rho > a$) the volume charge, that is compute the field everywhere.

2 A cylindrical region $\rho < a$ has a magnetic field that is decreasing in strength with time as $\vec{B} = \frac{\alpha}{t}\hat{z}$. A ring of wire with resistance R and radius $b < a$ lies in the $x-y$ plane centered at the origin.

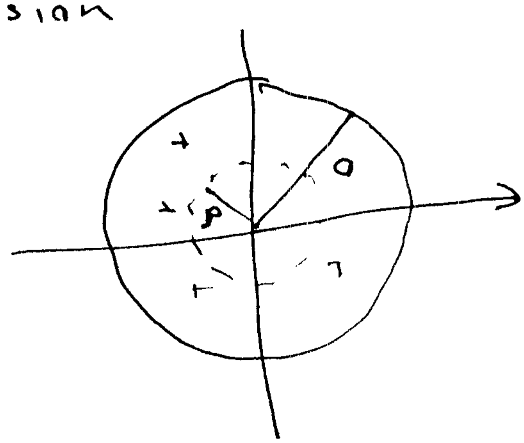
(a) Draw the system. Compute the current in the ring as a function of time. Indicate the direction of the current on your drawing.

(b) Compute the electric field caused by the changing magnetic field for $\rho < a$. Ignore any contributions of the wire.

Prb 1 A non-uniform cylindrical volume charge occupies the region $\rho < a$. The volume charge density varies as $\Gamma(\rho) = \gamma \rho^2$. Compute the field inside ($\rho < a$) and outside ($\rho > a$) of the volume

Sln Use Gauss Law, for

$\rho < a$ a cylindrical Gaussian surface of length l and radius ρ encloses a charge



$$Q = \int \Gamma \, dv$$

$$= l \int_0^{\rho} 2\pi\rho \, d\rho \, \Gamma = l \, 2\pi\gamma \int_0^{\rho} \rho^3 d\rho$$

$$= \frac{\pi \gamma \rho^4 l}{2}$$

The flux out of the Gaussian surface is

$$\Phi = 2\pi\rho l E = \frac{Q_{enc}}{\varepsilon_0} \quad \text{by Gauss}$$

$$2\pi\rho l E = \frac{\pi\gamma\rho^4 l}{2\varepsilon_0}$$

$\rho < a$

$$\vec{E} = \frac{\gamma \rho^3}{4\varepsilon_0}\,\hat{\rho}$$

$\rho > a$ $\qquad Q_{enc} = \dfrac{\pi \gamma a^4 l}{2}$

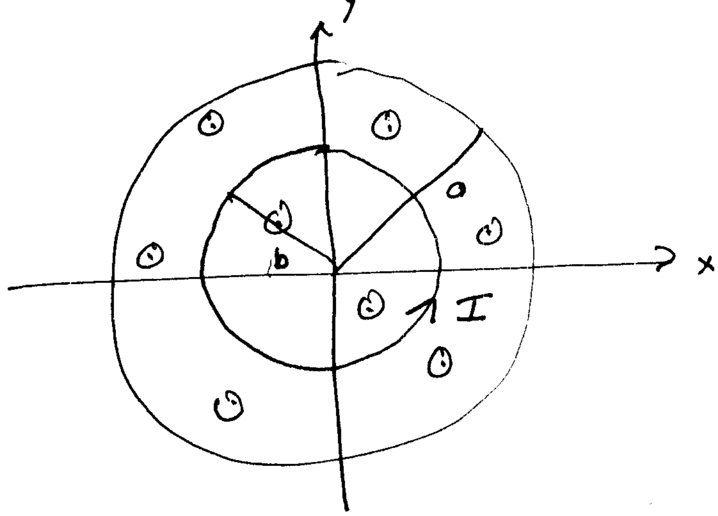
$$\vec{E} = \frac{\pi \gamma a^4 l}{4\pi \rho l \varepsilon_0}\,\hat{\rho} = \frac{\gamma a^4}{4\varepsilon_0 \rho}\,\hat{\rho}$$

<u>Prb 2</u> A cylindrical region $\rho < a$ has a magnetic field whose strength is decreasing as $\vec{B} = \frac{\gamma}{t}\hat{z}$

A ring of wire with resistance R and radius $b < a$ lies in the x-y plane, centered at the origin as shown

(a) Compute the ~~emf in the around the~~ current in the ring. Report both magnitude and direction

(b) Compute the electric field, if one exists, for $\rho < a$. Report field as vector.



(a) The direction of current is such that the flux due to the current opposes the change in ~~curr~~ flux, by RHR the current is counterclockwise.

Magnetic flux $\Phi_m = NBA = \frac{\pi b^2 \gamma}{t}$

$$\text{emf} = -\frac{d\Phi_m}{dt} = \frac{\pi b^2 \gamma}{t^2} \qquad I = \frac{\text{emf}}{R}$$

(b) Now compute the emf around a path of radius ρ.

$$\Phi_m = \frac{\pi \rho^2 \gamma}{t}$$

$$-\dot{\Phi}_m = \frac{\pi \rho^2 \gamma}{t^2} = \text{emf}$$

$$\text{emf} \equiv \oint \vec{E} \cdot d\vec{l} = 2\pi\rho\, E(\rho)$$

$$\vec{E}(\rho) = \frac{\pi \rho^2 \gamma}{2\pi\rho\, t^2}\,\hat{\phi} \qquad \text{(Direction from drawing)}$$

$$= \frac{\rho\gamma}{2t^2}\,\hat{\phi}$$

PHYS 3414 - Electricity and Magnetism- Test 1 - Part 2

All problems are worth 25 points. The majority of points on each problem will be awarded for doing the physics correctly; if you have correctly done the physics, but cannot carry out the mathematics, you will still receive most of the points.

3 For each of the following fields, determine if the field is a possible electromagnetic field.

(a) $\vec{E} = \gamma r^2 \hat{r}$ in spherical coordinates

(b) $\vec{B} = \gamma \rho^2 \hat{\rho}$ in cylindrical coordinates

If the field is a possible electromagnetic field, report the charge density Γ, the current density $\vec{J}$, and any additional field ($\vec{E}$ or $\vec{B}$) required. Some or all of these quantities may be zero.

4 A square of charge with uniform charge density σ lies in the $x - y$ plane, centered at the origin. The square has side of length $2a$. Calculate the field at a point a distance $R > a$ along the y axis.

Prb 3 For each of the following fields determine if they are a possible electromagnetic field.

(a) $\vec{E} = \gamma_0 r^2 \hat{r}$ $+ \cancel{\hat{\theta}}$ (spherical)

(b) $\vec{B} = \gamma_0 \rho^2 \hat{\rho}$ $\cancel{+\gamma_1 \hat{\phi}}$ (cylindrical coordinates)

If the field is possible, report the charge density Γ, current density $\vec{J}$ and the associated electric or magnetic field that is required.

(a) Check Maxwell

$$\nabla \cdot \vec{E} = \frac{1}{r^2}\frac{\partial r^2 A_r}{\partial r} = \Gamma/\epsilon_0$$

$$= \frac{\gamma_0}{r^2}\frac{\partial r^4}{\partial r} = \frac{4r^3\gamma_0}{r^2} = 4r\gamma_0 = \Gamma/\epsilon_0$$

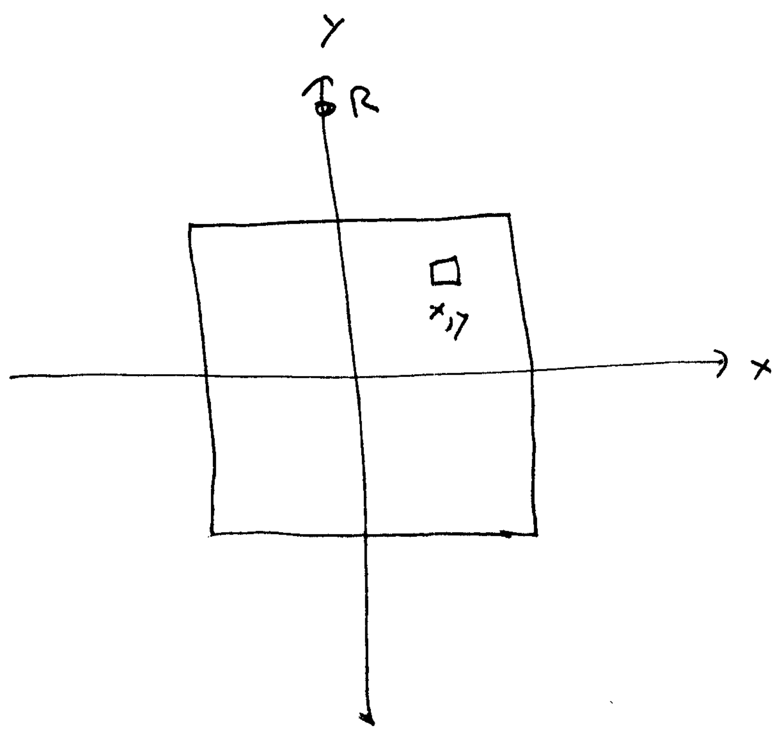
$\Gamma = 4r\gamma_0\epsilon_0$ charge density

$\nabla \times \vec{E} = 0$ ✓ $\vec{B} = 0$ or constant.

$\vec{J} = 0$ Possible field

(b) $\nabla \cdot \vec{B} \neq 0$ Not possible.

Prb 4 A square of charge 2a on side with surface charge density σ.



$$\vec{r} = (0, R, 0) \qquad \vec{r}\,' = (x', y', 0)$$

$$\vec{r}\,'' = \vec{r} - \vec{r}\,' = (-x', R-y', 0)$$

$$r'' = \sqrt{x'^2 + (R-y')^2}$$

$$\vec{E} = \int \frac{k\sigma\, dx'\, dy'\, \vec{r}\,''}{r''^3}$$

By symmetry the field is in y direction, take y component

$$\vec{E} = \hat{y} k\sigma \int_{-a}^{a} \int_{-a}^{a} \frac{(R-y')\,dx'\,dy'}{\left(x'^2 + (R-y')^2\right)^{3/2}}$$

$$u = R - y' \qquad du = -dy'$$

$$\vec{E} = -\hat{y} k\sigma \int_{-a}^{a} dx' \int_{R+a}^{R-a} \frac{u\,du}{\left(x'^2 + u^2\right)^{3/2}}$$

$$= -\hat{y} k\sigma \int_{-a}^{a} dx' \left[-\frac{1}{\sqrt{u^2+x^2}} \right]_{R+a}^{R-a}$$

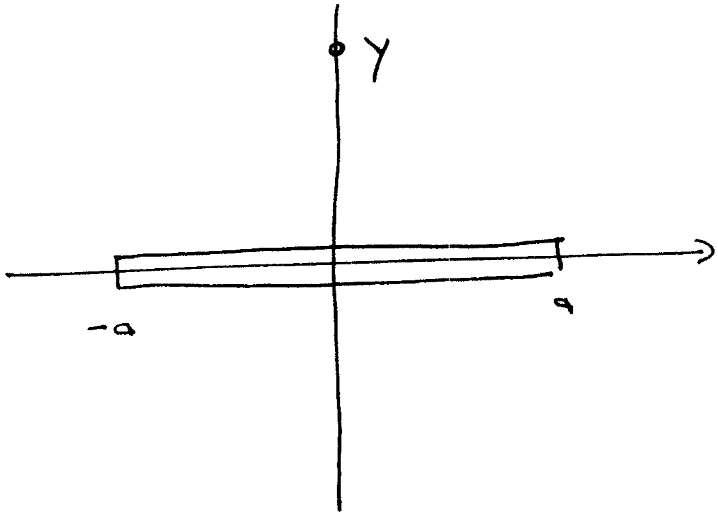
$$= -\hat{y} k\sigma \int_{-a}^{a} dx' \left[\frac{1}{\sqrt{(R+a)^2 + x'^2}} - \frac{1}{\sqrt{(R-a)^2 + y'^2}} \right]$$

$$= -\hat{y}\, 2k\sigma \int_{0}^{a} dx' \left[\frac{1}{\sqrt{(R+a)^2 + x'^2}} - \frac{1}{\sqrt{(R-a)^2 + y'^2}} \right]$$

$$= -\hat{y}\, 2k\sigma \left[\ln\left(x + \sqrt{x^2 + (R+a)^2}\right) - \ln\left(x + \sqrt{x^2 + (R-a)^2}\right) \right]_0^a$$

$$= 2\sigma k \hat{y} \left[\ln\left(\frac{a + \sqrt{a^2 + (R-a)^2}}{|R-a|} \right) - \ln\left(\frac{a + \sqrt{a^2 + (R+a)^2}}{|R+a|} \right) \right]$$

Method II – Compute field of line charge first.



$\vec{r}\,' = (x', 0, 0) \qquad \vec{r}\,(0, y, 0)$

$\vec{r}\,'' = (-x', y, 0)$

$$\vec{E} = \lambda k \int_{-a}^{a} \frac{dx' \, \vec{r}\,''}{r''^3}$$

$$E_y = \lambda k y \int_{-a}^{a} \frac{dx'}{\left(\sqrt{x'^2 + y^2}\right)^3}$$

$$= 2\lambda k y \left. \frac{x'}{y^2\sqrt{x'^2+y^2}} \right|_0^a$$

$$= \frac{2\lambda k y a}{y^2\sqrt{y^2+a^2}} = \frac{2\lambda k}{y}\left(\frac{a}{\sqrt{y^2+a^2}}\right)$$

Correct as $a \to \infty$

Now build square out of sequence of strips

$$\vec{E}_{square} = \int_{-a}^{a} \vec{E}_{strip}(R-y')\,dy' \qquad \lambda = \sigma\,dy'$$

$$\vec{E}_{square} = 2\sigma k \hat{y}\, a \int_{-a}^{a} \frac{dy'}{(R-y')\sqrt{(R-y')^2+a^2}}$$

$$u = R-y' \qquad du = \cancel{-R_y} = -dy'$$

$$\vec{E}_{square} = -2\sigma k a \hat{y} \int_{R+a}^{R-a} \frac{du}{u\sqrt{u^2+a^2}}$$

$$= -2k\sigma a \hat{y}\left(-\frac{1}{a}\ln\left(\frac{a+\sqrt{u^2+a^2}}{u}\right)\right)\Bigg|_{R+a}^{R-a}$$

$$= -2k\sigma \hat{y}\left(\ln\left(\frac{a+\sqrt{(R-a)^2+a^2}}{R-a}\right)\right.$$

$$\left. - \ln\left(\frac{a+\sqrt{(R+a)^2+a^2}}{R+a}\right)\right)$$

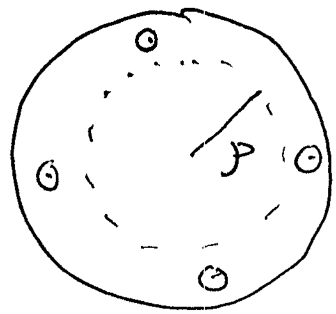
PHYS 3414 - Electricity and Magnetism- Test 1 - Conflict

All problems are worth 25 points. The majority of points on each problem will be awarded for doing the physics correctly; if you have correctly done the physics, but cannot carry out the mathematics, you will still receive most of the points. Final tournament score : Woodland 25 - Bentonville Black 17. Woodland finishes season as undefeated regional champions. I wrote the test after the tournament.

C1 A non-uniform semi-infinite cylindrical current with volume current density $\vec{J} = \gamma\rho^2\hat{z}$ occupies the region $\rho < a$. Compute the field inside ($\rho < a$) and the field outside ($\rho > a$) the current, that is compute the field everywhere.

C2 A spherical region $r < a$ contains a non-uniform volume charge $\Gamma = \gamma r^3$ where γ is a constant. Compute the field everywhere.

Conflict 1 - $\vec{J} = \gamma \rho^2 \hat{z}$ for $\rho < a$



Use circular Amperian path of radius ρ. The current encircled by the path is

$$I_{enc} = \int \vec{J} \cdot da = \int_0^{\rho} 2\pi\rho \, d\rho \, J$$

$$= \int_0^{\rho} 2\pi \gamma \rho^3 d\rho$$

$$= \frac{\pi \gamma \rho^4}{2}$$

Ampere's Law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} = 2\pi\rho \, B(\rho)$$

$$\vec{B}(\rho) = \frac{\mu_0 I_{enc}}{2\pi\rho} \hat{\phi}$$ — Direction found with RHR

Inside Current $(\rho < a)$

$$\vec{B} = \frac{\mu_0 I_{enc}}{2\pi\rho}\hat{\phi} = \frac{\mu_0 \pi \gamma \rho^4/2}{2\pi\rho}\hat{\phi}$$

$$= \frac{\mu_0 \gamma \rho^3}{4}\hat{\phi}$$

Outside Current $\rho > a$

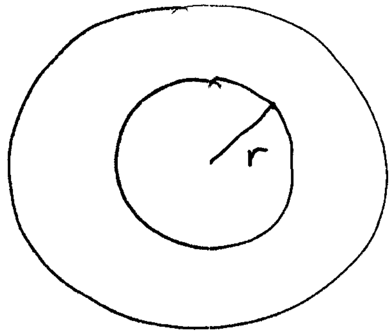
$$I_{enc} = \frac{\pi\gamma a^4}{2}$$

$$\vec{B} = \frac{\mu_0 I_{enc}}{2\pi\rho}\hat{\phi} = \frac{\mu_0 \pi \gamma a^4/2}{2\pi\rho}\hat{\phi}$$

$$= \frac{\mu_0 \gamma a^4}{4\rho}\hat{\phi}$$

Since $[\gamma \rho^2] = A/m^2$ $\quad [\gamma a^4] = \text{Amps}$

Conflict 2 $\qquad$ Spherical volume $\Gamma = \frac{\gamma}{r^2}$ $\quad r<a$



Use a spherical Gaussian surface of radius r.

If $r<a$, then the charge enclosed is

$$Q_{enc} = \int_0^r 4\pi r^2 dr\, \Gamma \;\cancel{= 4\pi\Gamma \int_0^r \frac{dr}{r^2}}$$

$$= \int_0^r 4\pi r^5 dr\, \gamma$$

$$= 4\pi\gamma \frac{r^6}{6}$$

The flux out of the Gaussian surface is

$$\Phi = EA = 4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_0} \quad \text{by Gauss' Law}$$

$$\vec{E} = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{4\pi\gamma r^6/6}{4\pi\varepsilon_0 r^2}\hat{r} \qquad r<a$$

$$= \frac{r^4\gamma}{6\varepsilon_0}\hat{r} \qquad r<a$$

If $r > a$, $Q_{enc} = 4\pi\gamma a^6/6$

$$\vec{E} = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{4\pi\gamma a^6/6}{4\pi\varepsilon_0 r^2}\hat{r}$$

$$= \frac{\gamma a^6}{6\varepsilon_0 r^2}\hat{r}$$