PHYS 3414 - Electricity and Magnetism- Test 2 - Part 1

All problems are worth 25 points. The majority of points on each problem will be awarded for doing the physics correctly; if you have correctly done the physics, but cannot carry out the mathematics, you will still receive most of the points. I write the test after we won the soccer game.

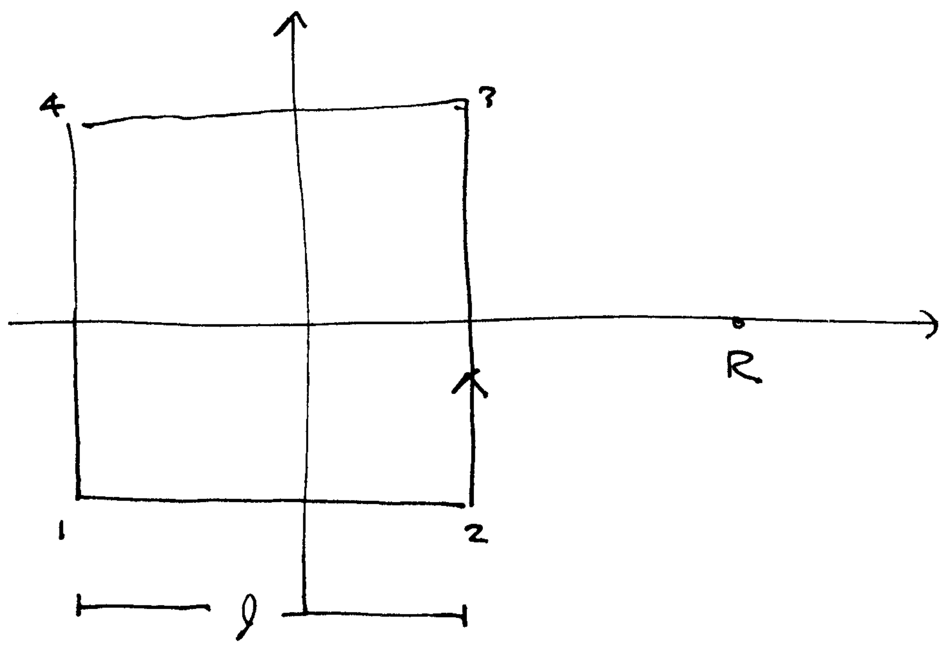
1 A flat square loop of wire with side length ℓ in in the x-y plane centered at the origin. The loop carries a current I in the clockwise direction when viewed from the positive z axis. Compute the vector potential at a point a distance $R > \ell$ along the x axis.

2 A spherical capacitor is formed of two conductors of radius a and b where $a < b$. The capacitor is centered at the origin. Half the capacitor ($z < 0$) is filled with a dielectric with relative permittivity ϵ_1 and half the capacitor ($z > 0$) with dielectric with relative permittivity ϵ_2.

a Compute the capacitance.

b Compute the total charge stored on the inner conductor if a potential V_0 is established across the two conductors. Report the division of this charge between the top half, Q_+ where $z > 0$; and Q_- where $z < 0$.

1



$$\vec{A} = \vec{A}_{12} + \vec{A}_{23} + \vec{A}_{34} + \vec{A}_{41}$$

$$\vec{A}_{12} + \vec{A}_{34} = 0 \quad \text{by symmetry}$$

$$\vec{A}_{12} = \frac{\mu_0}{4\pi}\int \frac{I\,d\vec{l}}{r''} \qquad r'' = \sqrt{y^2 + (R - l/2)^2}$$

Let $R - l/2 = d_{12R}$

$$= \frac{\mu_0 I}{4\pi}\hat{y}\int_{-l/2}^{l/2} \frac{dy}{\sqrt{y^2 + d_{12R}^2}}$$

$$= \frac{\mu_0 I}{2\pi}\hat{y}\int_{0}^{l/2} \frac{dy}{\sqrt{y^2 + d_{12R}^2}}$$

$$\int \frac{dy}{\sqrt{y^2+a^2}} = \ln\left(y + \sqrt{y^2+a^2}\right) \quad \text{Schaum's}$$

$$\vec{A}_{12} = \frac{\mu_0 I}{2\pi}\hat{y} \ln\left(\frac{l/2 + \sqrt{(l/2)^2 + d_{12R}^2}}{d_{12R}}\right)$$

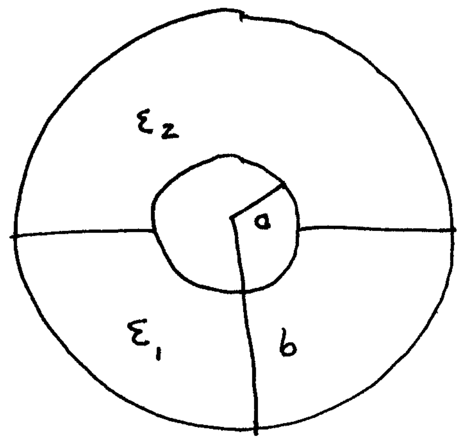
$$\vec{A}_{41} = -\frac{\mu_0 I}{2\pi}\hat{y} \ln\left(\frac{l/2 + \sqrt{(l/2)^2 + d_{41R}^2}}{d_{41R}}\right)$$

$$d_{41R} = R + l/2$$

$$\vec{A} = \vec{A}_{12} + \vec{A}_{41}$$

$$= \frac{\mu_0 I}{2\pi}\hat{y} \ln\left(\left(\frac{R + l/2}{R - l/2}\right) \frac{l/2 + \sqrt{(l/2)^2 + (R - l/2)^2}}{l/2 + \sqrt{(l/2)^2 + (R + l/2)^2}}\right)$$

2



The spherical shell capacitor with air filling the capacitor has capacitance:

Place Q on the inner conductor

$$\vec{E} = \frac{kQ}{r^2}\hat{r}$$

$$\Delta V = -\int \vec{E}\cdot d\vec{l} = \left.\frac{kQ}{r}\right|_a^b$$

$$= kQ\left(\frac{1}{b} - \frac{1}{a}\right)$$

$$C = \frac{Q}{|\Delta V|} = \frac{Q}{kQ\left|\left(\frac{1}{b} - \frac{1}{a}\right)\right|} = \frac{4\pi\varepsilon_0}{\frac{1}{a} - \frac{1}{b}}$$

The capacitance with a dielectric completely filling the fieldspace is

$$C_1 = \varepsilon_1 C$$

$$C_2 = \varepsilon_2 C$$

The capacitance of half a capacitor is half the capacitance and the top and bottom ~~[illegible]~~ halves are connected is parallel

$$C_t = \frac{1}{2} C_1 + \frac{1}{2} C_2$$

$$= \frac{2\pi\varepsilon_0 \varepsilon_1}{\left(\frac{1}{a} - \frac{1}{b}\right)} + \frac{2\pi\varepsilon_0 \varepsilon_2}{\left(\frac{1}{a} - \frac{1}{b}\right)}$$

$$= \frac{2\pi\varepsilon_0 (\varepsilon_1 + \varepsilon_2)}{\frac{1}{a} - \frac{1}{b}}$$

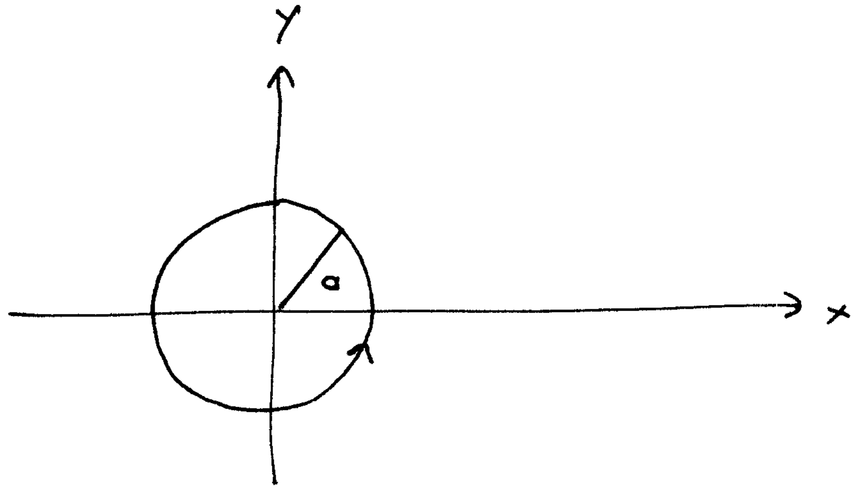
(b)

$$Q_+ = \frac{1}{2} C_1 V_0 = \frac{2\pi \varepsilon_0 \varepsilon_1 V_0}{\frac{1}{a} - \frac{1}{b}}$$

$$Q_- = \frac{1}{2} C_2 V_0 = \frac{2\pi \varepsilon_0 \varepsilon_2 V_0}{\frac{1}{a} - \frac{1}{b}}$$

$$Q_+ = C_+ V_0 = \frac{2\pi \varepsilon_0 V_0 (\varepsilon_1 + \varepsilon_2)}{\frac{1}{a} - \frac{1}{b}}$$

(3)



The bound current density $\vec{K}_b = \vec{M} \times \hat{n} = M_o \hat{\phi}$

The total bound current is $I = K_b d = M_o d = 100A$.

The field at the center is found by integrating the current around the loop

$$\vec{B} = \oint_c d\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times \hat{r}}{a^2}$$

$d\vec{l} = a\, d\phi\, \hat{\phi}$

$\hat{r} = -\hat{r}$

$\hat{\phi} \times -\hat{r} = \hat{z}$

$$= \frac{\mu_0 I}{4\pi a^2} \hat{z} \int_0^{2\pi} a\, d\phi$$

$$= \frac{\mu_0 I}{4\pi a} \cdot 2\pi \hat{z} = \frac{\mu_0 I}{2a} \hat{z} = \frac{\mu_0 M_o d}{2a} \hat{z}$$

$$= \frac{\left(4\pi \times 10^{-7} \frac{Tm}{A}\right)(100A)}{2(0.01m)} \hat{z} = 2\pi \times 10^{-3}\, T \hat{z}$$

(b) The moment is $\vec{m} = M_0 V \hat{z} = M_0 \pi a^2 l \hat{z}$

where I used the right-hand rule for the moment to get the direction.

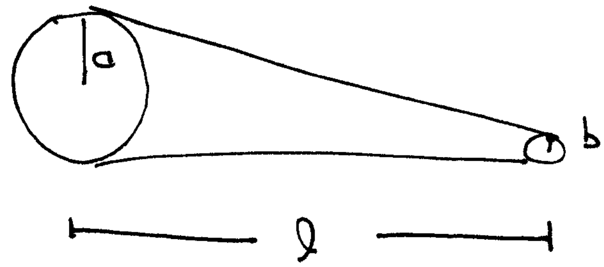
$$\vec{\tau} = \vec{m} \times \vec{B} = B_0 M_0 \pi a^2 l \, \hat{z} \times \hat{x}$$

$$= B_0 M_0 \pi a^2 l \, \hat{y}$$

$$= \left(\tfrac{1}{5} T\right)\left(10^5 \tfrac{A}{m}\right)(\pi)(0.01m)^2(0.001m) \, \hat{y}$$

$$= \frac{\pi}{5} \times 10^{-2} \hat{y} = 0.00628 \, Nm \, \hat{y}$$

(4)



(a) The current I crossing any cross-section must be the same. Let x be the distance along the wire.

The current density is then

$$J(x) = \frac{I}{\text{Area}} = \frac{I}{\pi r(x)^2}$$

The radius is given by $r(x) = a + \frac{(b-a)}{\ell} x \equiv a + \gamma x$

$$J(x) = \frac{I}{\pi (a+\gamma x)^2}$$

The field is related to the current by Ohm's law,

$$J = \sigma E(x) = \frac{E(x)}{\rho}$$

$$E(x) = \rho J = \frac{\rho I}{\pi (a+\gamma x)^2}$$

The potential difference between the ends is

$$\Delta V = -\int_0^{\ell} E(x)\,dx = -\frac{\rho I}{\pi}\int_0^{\ell}\frac{dx}{(a+\gamma x)^2}$$

$$u = a+\gamma x \qquad du = \gamma\,dx$$

$$\Delta V = -\frac{\rho I}{\gamma\pi}\int_a^b \frac{du}{u^2} = \frac{\rho I}{\gamma\pi}\left(\frac{1}{b}-\frac{1}{a}\right) = V_0$$

$$R = \frac{V_0}{I} = \frac{\rho}{\gamma\pi}\left|\left(\frac{1}{b}-\frac{1}{a}\right)\right| = \frac{\rho\ell}{(b-a)\pi}\left|\left(\frac{1}{b}-\frac{1}{a}\right)\right|$$

$$= \frac{\rho\ell}{\pi ab}$$

(b) The field is

$$\vec{E} = \frac{\rho I}{\pi(a+\gamma x)^2}\,\hat{x} \qquad \text{eliminate } I$$

$$I = \frac{\gamma\pi V_0}{\rho\left(\frac{1}{b}-\frac{1}{a}\right)}$$

$$\vec{E} = \frac{\gamma V_0}{\left(\frac{1}{b}-\frac{1}{a}\right)(a+\gamma x)^2}$$