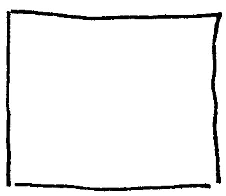
Boundary Value Review

1D Rectangular $V(x)$

Solution $V(x) = A + Bx$

BC $V(0) = V_1$ $V(a) = V_2$

2D Rectangular $V(x, y)$



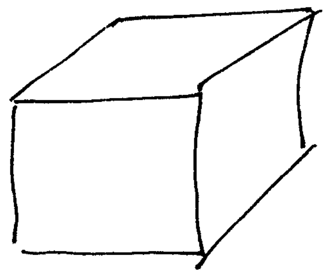
Solution $[e^{kx}, e^{-kx}] \times [\sin ky, \cos ky]$

or

$[\sin kx, \cos kx] \times [e^{ky}, e^{-ky}]$

B.C. V on boundary – Any V can be applied to boundary.

3D Rectangular



Solution

$$\left[e^{k_1 x},\ e^{-k_1 x}\right] \times \left[\sin k_2 y,\ \cos k_2 y\right] \times \left[\sin k_3 z,\ \cos k_3 z\right]$$

$$k_1^2 - k_2^2 - k_3^2 = 0$$

$$\left[e^{k_1 x},\ e^{-k_1 x}\right] \times \left[e^{k_1 y},\ e^{-k_1 y}\right] \times \left[\sin k_3 z,\ \cos k_3 z\right]$$

$$k_1^2 + k_2^2 - k_3^2 = 0$$

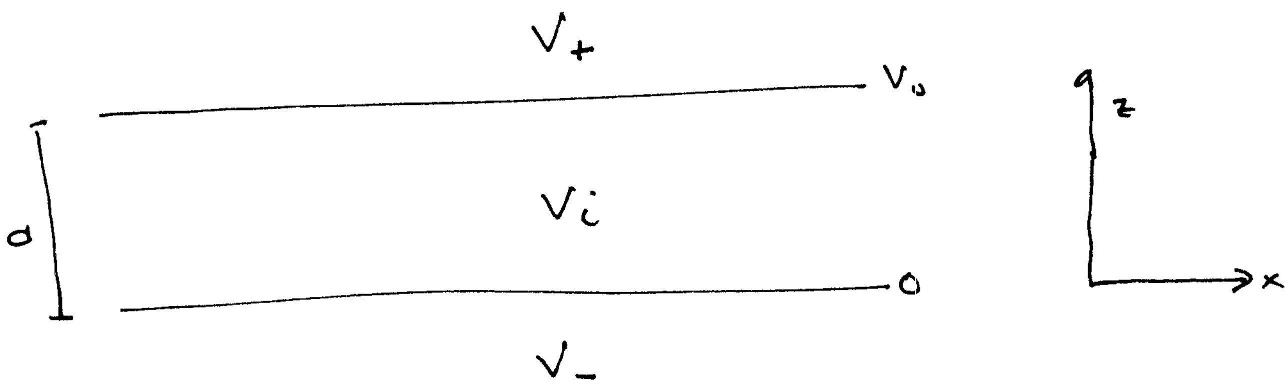
BC. V on boundary.

B Rectangular Trivial Solutions

x, y, z

$\Rightarrow$ Used in Gauss' Law Geometries.

Ex



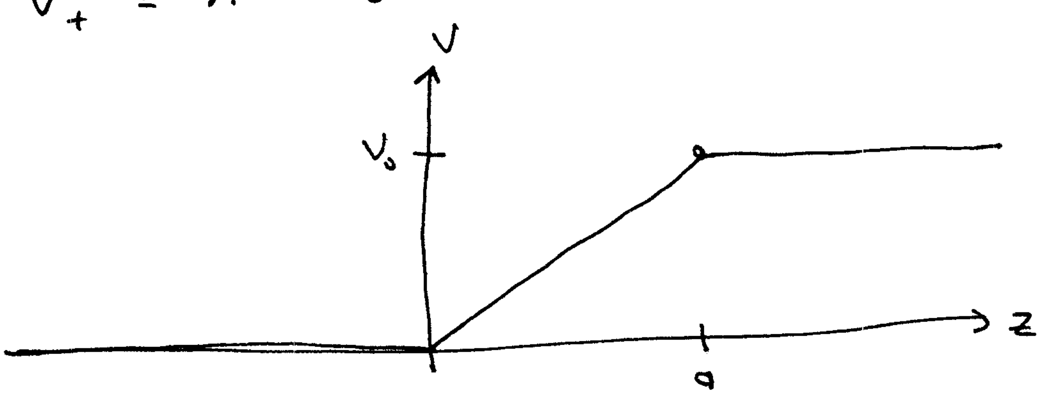
Two infinite parallel Planes

$$V_i(z) = A + Bz = \frac{V_0}{a} z$$

Outside the planes If V finite as $z \to \pm\infty$,

$$V_- = A = 0$$

$$V_+ = A = V_0$$



Electric Field

$$\vec{E}_+ = 0 \qquad \vec{E}_- = 0$$

$$\vec{E}_i = -\frac{V_0}{d}\,\hat{z}$$

Charge

$$\sigma_{top} = \varepsilon_0\left[(\vec{E}_+ \cdot \hat{n}) - \vec{E}_i \cdot \hat{n}\right]$$

$$= \varepsilon_0 \frac{V_0}{d}$$

$$\sigma_{bottom} = \varepsilon_0\left[\vec{E}_i \cdot \hat{n} - \vec{E}_- \cdot \hat{n}\right]$$

$$= -\frac{\varepsilon_0 V_0}{a}$$

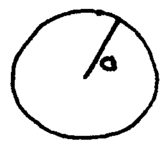
Capacitance / Area

$$C = \frac{\sigma_t A}{V_0} = \frac{\varepsilon_0 A}{a}$$

$$C/A = \frac{\varepsilon_0}{a}$$

Cylindrical Systems

(1) z independent (infinite cylinders)



B.C. $V(a,\phi)$ on surface

— or —

$\vec{E} = E_0 \hat{x}$ at a large distance

with conducting or dielectric cylinder.

— or —

$\sigma(\phi)$ given on surface.

Solutions

$\rho^n \cos n\phi \qquad \rho^n \sin n\phi$

$\rho^{-n} \cos n\phi \qquad \rho^{-n} \sin n\phi$

Orthogonality

$$\int_0^{2\pi} \cos n\phi \cos m\phi \, d\phi = \pi \delta_{nm} \qquad n > 0$$

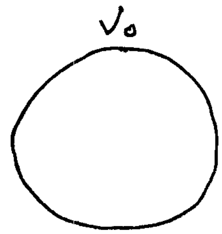
$$\int_0^{2\pi} \cos 0\phi \, \cos m\phi \, d\phi = 2\pi \delta_{0m}$$

$$\int_0^{2\pi} \sin n\phi \, \sin m\phi \, d\phi = \pi \delta_{nm}$$

Trivial Solutions

$$V(x,y,z) = 1 = V(\rho, \phi, z)$$

Ex



Find potential inside.

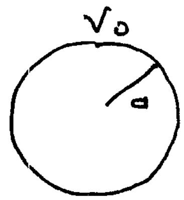
$$\cancel{V(\rho, a) = V_0 \Longrightarrow V(\rho, \phi}$$

$$V(a, \phi) = V_0 \Longrightarrow V_i(\rho, \phi) = V_0$$

Trivial Solution II

$$V(\rho, \phi, z) = \ln \rho$$

Ex $\quad V(a, \phi) = V_0 \quad$ Find potential outside

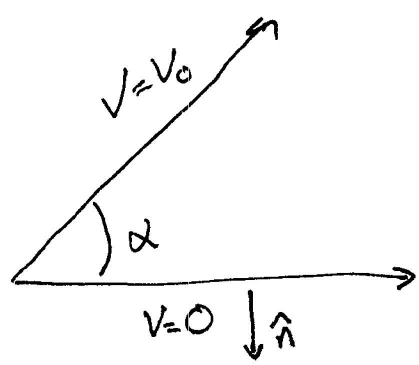


$$V = A \ln \rho = V_0 \ln\left(\rho/a\right)$$

Trivial Solution III

$$V(\rho, \phi, z) = \phi$$

Ex $\quad$ Wedge held at constant potential



$$V(\rho, \phi, z) = V_0 \phi/\alpha$$

Compute Field

$$\vec{E} = -\nabla V = -\frac{V_0}{\rho\alpha}\hat{\phi}$$

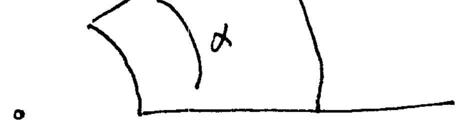
Compute Charge (Assume wedge formed of conducting plates no field outside).

V_0 plane $\quad (E_{outside} \cdot \hat{n}) - (E_{inside} \cdot \hat{n}) = \frac{\sigma}{\varepsilon_0}$

$$0 - \frac{V_0}{\rho\alpha} = \frac{\sigma}{\varepsilon_0}$$

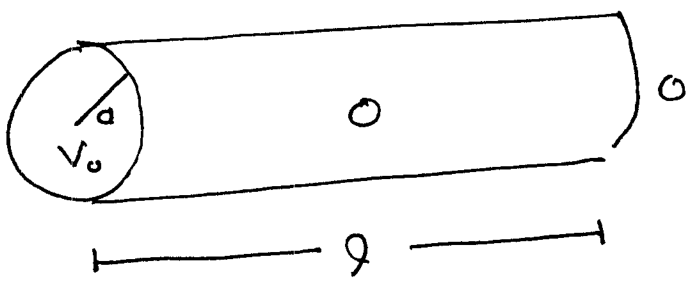
$$\sigma = \frac{-V_0 \varepsilon_0}{\rho\alpha}$$

Compute Capacitance This cannot be done since $\rho \to \infty$ but it can be done if we take geometry where origin is excluded.



Cylindrical Systems – z dependant

<u>Case I</u> Ends fixed potential, sides of cylinder zero.



<u>Solutions</u>

$$\begin{bmatrix} \text{Bessel Functions} \\ J_\nu, N_\nu \end{bmatrix} \times \left[\sin \nu\phi, \cos \nu\phi\right]$$

$$\times \left[e^{kz}, e^{-kz}\right]$$

Covered in lecture, not on test.

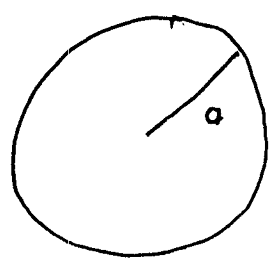
Case II Ends at zero potential, potential varies on side.

Solutions

$$\left[\text{Modified Bessel Functions}\right] \times \left[\sin \nu\phi,\ \cos \nu\phi\right]$$
$$\times \left[\sin kz,\ \cos kz\right]$$

Not covered

Spherical Systems (ϕ independant)



Solutions $r^n P_n(\cos\theta)$ $r^{-(n+1)} P_n(\cos\theta)$

BC $V(r,\theta)$ given on sphere

$\sigma(\theta)$ given on sphere

conducting or dielectric sphere placed in field the varies as $\vec{E} = E_0 \hat{x}$ at long range.

Orthogonality

Good

$$\int_{-1}^{1} P_m(\cos\theta)\,P_n(\cos\theta)\,d(\cos\theta) = \frac{2}{2n+1}\,\delta_{nm}$$

Griffith's

$$\int_{0}^{\pi} P_n(\cos\theta)\,P_m(\cos\theta)\,\sin\theta\,d\theta = \frac{2}{2n+1}\,\delta_{nm}$$

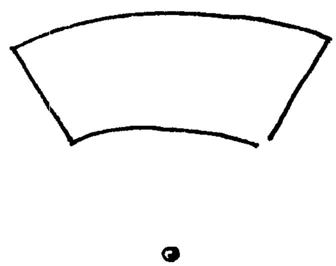
Trivial Solutions

$V = 1 \implies$ Potential inside sphere at constant

$$V(a,\theta) = V_0$$

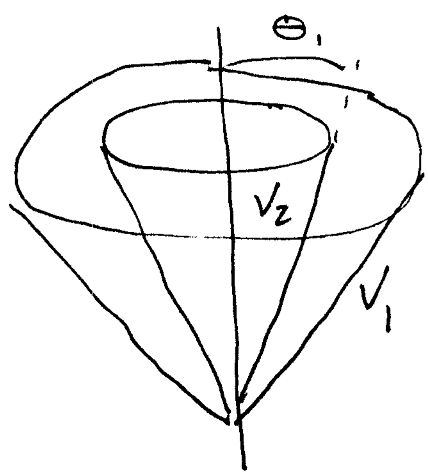
$V = \frac{1}{r} \implies$ Potential outside sphere at constant V_0.

$V = \phi \implies$ Potential defined on $V(\phi_1)$ and $V(\phi_2)$ sections of a sphere.



Same as cylindrical problem but now spread out over sphere.

$V = \ln\left(\tan \theta/2\right)$ - Potential defined on two $\theta =$ constant surfaces.



Spherical Systems - ϕ dependent

Not covered - spherical harmonics.