PHYS 3414 - Electricity and Magnetism- Test 3

All problems are worth 25 points. Turn in solutions to four of the six problems to be graded. If you turn in more than four solutions, I will grade the first four. You are allowed to drop one-half of a test, so I will take the first two problems turned in as the first half-test and the second two problems turned in as the second half-test.

1 An infinite grounded conducting plane occupies the area $z < 0$. A dipole is formed of two charges $\pm Q$ separated by a distance a. The dipole moment is parallel to the z axis. The center or the dipole is a distance R from the plane. Compute the force on each charge and the total force on the dipole. Is the dipole attracted or repelled from the plane?

2 A grounded conducting sphere of radius a is centered at the origin. Two charges $+q$ are a distance $\pm D\hat{x}$ along the x-axis. Compute the electric potential at the point $2D\hat{x}$.

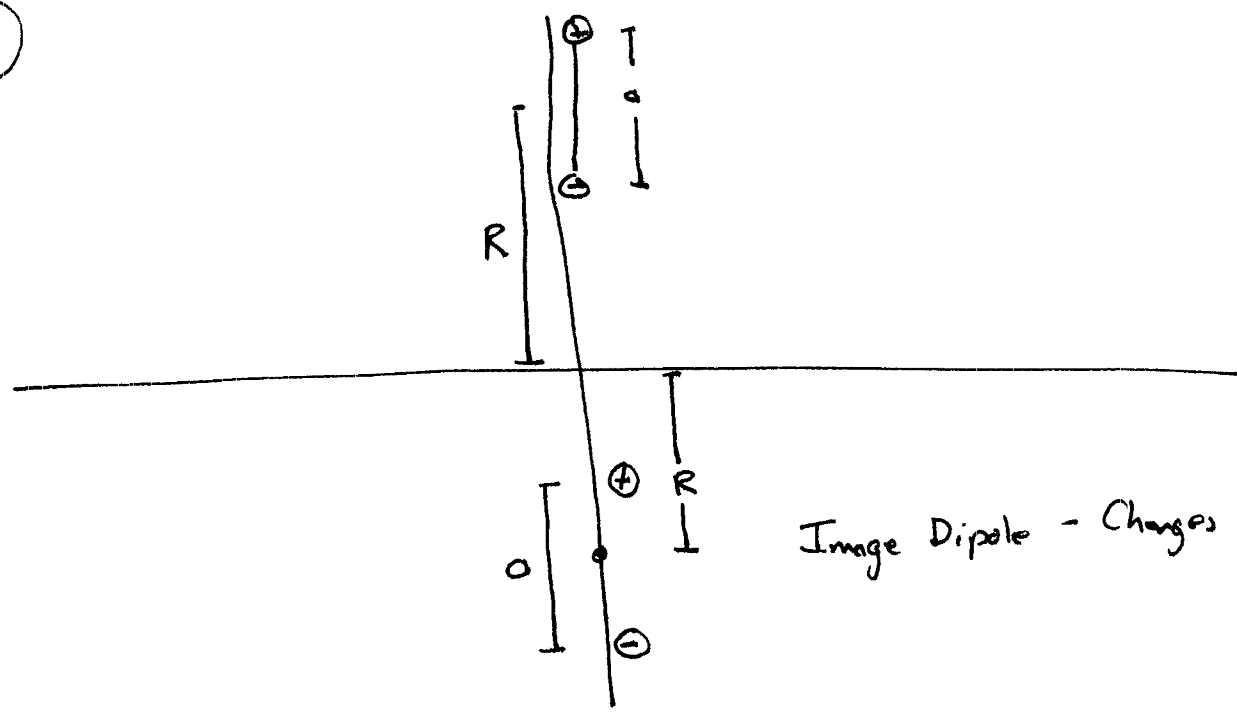
3 An infinite conducting cylinder of radius a is in an external electric field that is a uniform $E_0\hat{x}$ far from the cylinder. Compute the surface charge density as a function of ϕ on the surface of the cylinder.

4 A spherical system has a potential $V = V_0(1 - \cos^2(\theta))$ on the surface of sphere of radius a. Report the general expansion for the potential inside the sphere. Compute the first two coefficients of the expansion.

5 The potential of a rectangular system is independent of z. The system extends to infinity in the x direction. On the plane $y = 0$ and $y = a$ the potential is zero. On the plane $x = 0$, the potential is $V(0, y) = V_0 \sin^2(\pi y/a)$. Compute the potential in the channel. Report the formula you would use to calculate the coefficients in the series.

6 The half-plane $y = 0$, $x > 0$ is at $V = 0$. The half-plane $x = 0$, $y > 0$ is at potential V_0. Compute the electric field in the region $x > 0$, $y > 0$.

①



The force exerted on the negative charge is

$$F_- = \frac{-kq^2}{(2R-a)^2} + \frac{kq^2}{(2R)^2}$$

] upward positive

and the force on the positive charge

$$F_+ = \frac{kq^2}{(2R)^2} - \frac{kq^2}{(2R+a)^2}$$

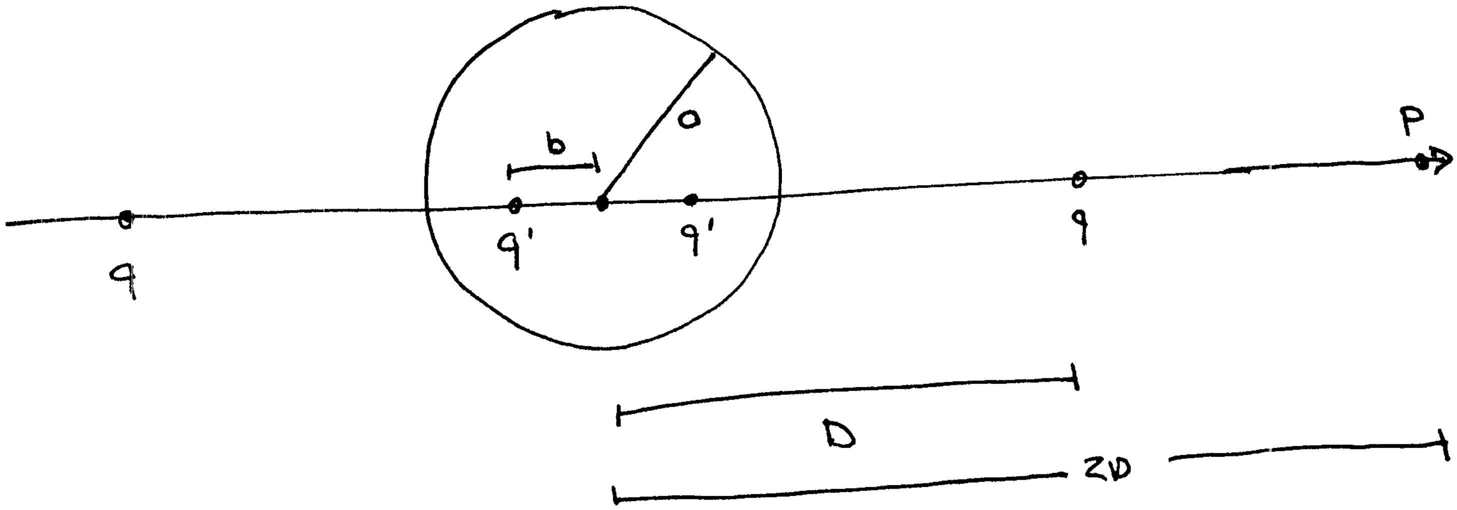
The total force exerted by the plane on the dipole is

$$F = F_- + F_+ = 2\cdot\frac{kq^2}{4R^2} - \frac{kq^2}{(2R+a)^2} - \frac{kq^2}{(2R-a)^2}$$

$$= \frac{kq^2a^2(a^2-12R^2)}{2R^2(2R+a)^2(2R-a)^2} < 0$$

You could also argue the force was attractive because the dipoles are anti-aligned.

2



To bring the sphere to zero potential use two image charges

$$q' = -\frac{a}{D}\,q$$

at $\qquad \pm b = \frac{a^2}{D}$

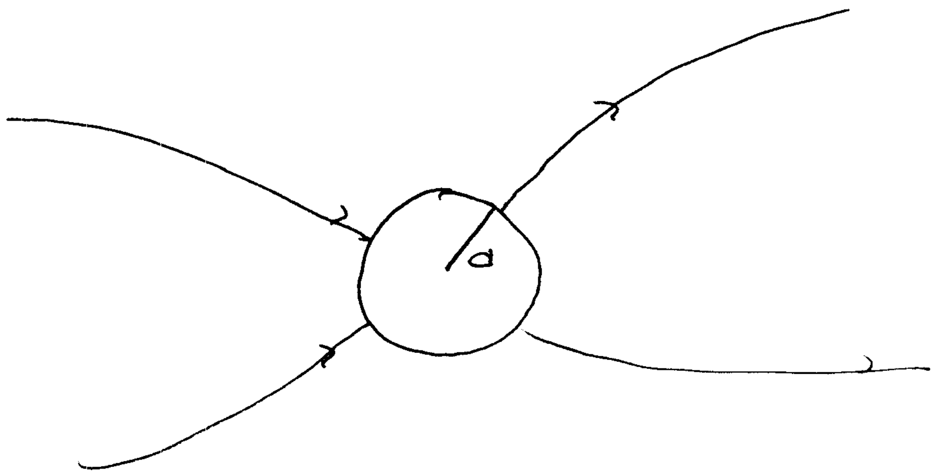
The total potential at P is the sum of the potentials.

$$V = \frac{kq}{3D} + \frac{kq'}{(b+2D)} + \frac{kq'}{\cancel{2D-2b})}\atop{(2D-b)} + \frac{kq}{D}$$

$$= \frac{4kq\,(a^4 - 4D^4 + 3aD^3)}{3D\,(a^4 - 4D^4)}$$

If you simplified, Maple says...

③



The correct solution to Laplace's Eqn is

$$V(\rho,\phi) = -E_0 \rho\cos\phi + \sum A_n \rho^{-n}\cos\phi$$

where the first term is to satisfy the boundary condition that $E \to E_0\hat{x}$ and we have discarded the sine solutions because they will not contribute because of the $\cos\phi$ boundary condition.

Let the surface of the sphere be the zero of potential.

$$V(a,\phi) = 0 \quad \Longrightarrow \quad A_n = 0 \quad n > 1$$

$$0 = -E_0 a\cos\phi + A_1 a^{-1}\cos\phi$$

$$A_1 = E_0 a^2$$

$$V(\rho,\phi) = -E_0 \rho \cos\phi + \frac{E_0 a^2}{\rho} \cos\phi$$

Use Gaussian pillbox to find the charge

$$\Phi = \vec{E}_0 \cdot \hat{\rho} A \underbrace{- \vec{E}_i \cdot \hat{\rho} A}_{0} = \frac{\sigma(\phi) A}{\varepsilon_0}$$

$$\vec{E}_0 \cdot \hat{\rho} = -\frac{\partial V_0}{\partial \rho}\bigg|_a = -\left(-E_0 \cos\phi - E_0 \frac{a^2}{a^2} \cos\phi\right)$$

$$= 2E_0 \cos\phi = \frac{\sigma(\phi)}{\varepsilon_0}$$

$$\sigma(\phi) = 2\varepsilon_0 E_0 \cos\phi$$

(4) $V(a,\theta) = V_0(1-\cos^2\theta)$

The solution to the Laplacian inside the sphere is

$$V_i(r,\theta) = \sum_n A_n r^n P_n(\cos\theta)$$

On the surface of the sphere

$$V_i(a,\theta) = V_0(1-\cos^2\theta) = \sum_n A_n a^n P_n(\cos\theta)$$

Use orthogonality

$$\int_{-1}^{1} V_i(a,\theta) P_m(\cos\theta)\, d\cos\theta$$

$$= \sum_n A_n a^n \int_{-1}^{1} P_n(\cos\theta) P_m(\cos\theta)\, d\cos\theta$$

$$= \frac{A_m a^m 2}{2m+1}$$

$$A_m = \frac{2m+1}{2a^m} \int_{-1}^{1} V_i(a,\theta) P_m(\cos\theta)\, d\cos\theta$$

Evaluate the first two terms

$$P_0 = 1$$

$$A_0 = \frac{1}{2}\int_{-1}^{1} V_0(1-\cos^2\theta)\cdot 1 \cdot d\cos\theta \qquad \text{Let } u = \cos\theta,\ du = d\cos\theta$$

$$= \frac{1}{2}V_0 \int_{-1}^{1} 1 - u^2\, du$$

$$= \frac{1}{2}V_0\left(u - \frac{u^3}{3}\right)\Big|_{-1}^{1}$$

$$= \frac{1}{2}V_0\left(2 - \frac{2}{3}\right) = \frac{2}{3}V_0$$

$$P_1 = x$$

$$A_1 = \frac{3}{2a}\int_{-1}^{1} V_0(1-\cos^2\theta)\cos\theta\, d\cos\theta$$

$$= \frac{3}{2a}\int_{-1}^{1} V_0(1-u^2)\,u\,du$$

$$= \frac{3V_0}{2a}\int_{-1}^{1}(u-u^3)\,du = 0$$

Alternatively, if you are tricky

$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$$

$$\frac{2P_2 + 1}{3} = \cos^2\theta$$

$$\frac{2}{3}P_2 + \frac{1}{3}P_0 = \cos^2\theta$$

$$1 - \cos^2\theta = P_0 - \cos^2\theta = P_0 - \frac{2}{3}P_2 - \frac{1}{3}P_0$$

$$= \frac{2}{3}(P_0 - P_2)$$

So our initial boundary condition reads,

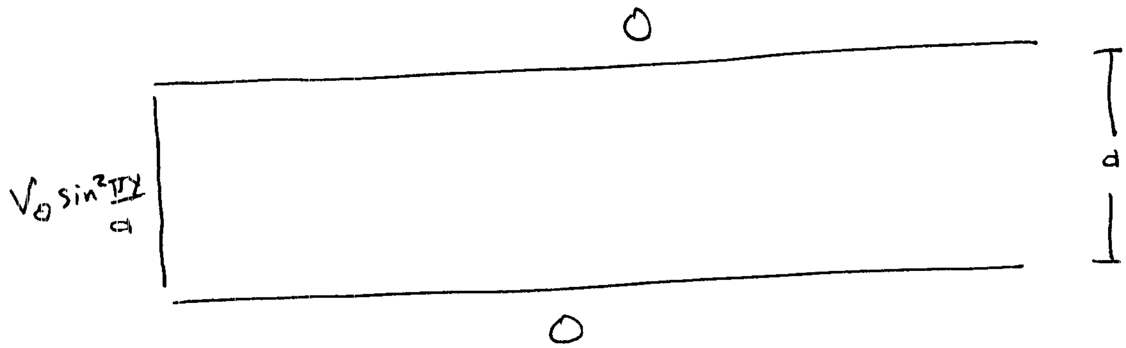
$$V_0(1-\cos^2\theta) = \frac{2}{3}\left(P_0(\cos\theta) - P_2(\cos\theta)\right)V_0$$

$$= \sum_n A_n a^n P_n(\cos\theta)$$

$$\frac{2}{3}V_0 = A_0 \qquad -\frac{2}{3}V_0 = A_2 a^2$$

All other terms zero.

(5)



Separated Laplacian

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

$$k^2 \qquad\qquad -k^2$$

The solutions are $[e^{kx}, e^{-kx}] \times [\sin ky, \cos ky]$

- To meet the y boundary condition, discard $\cos ky$ and set $k = \frac{n\pi}{a}$
- Discard e^{kx} because it blows up at ∞.

The general solution is then

$$V(x,y) = \sum A_n e^{-k_n x} \sin k_n y$$

At $x = 0$

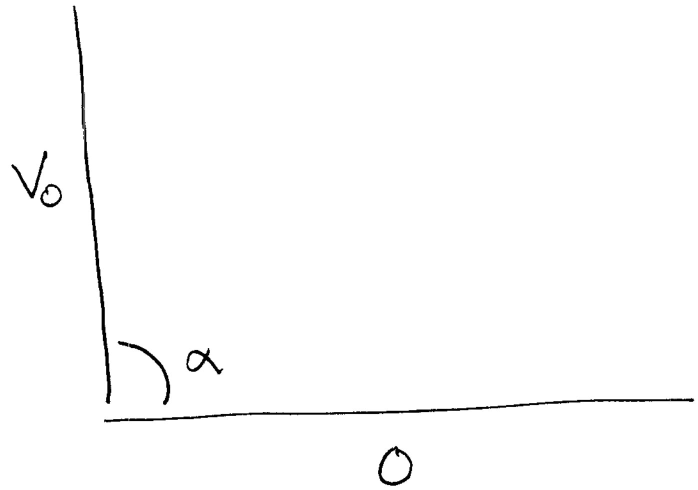
$$V(0,y) = V_0 \sin^2 \frac{\pi y}{a} = \sum A_n \sin k_n y$$

Use orthogonality $\displaystyle\int_0^a \sin k_n y \sin k_m y \, dy = \frac{a}{2} \delta_{nm}$

$$\frac{a}{2} A_m = V_0 \int_0^a \sin \frac{m\pi}{b} y \sin^2 \frac{\pi y}{a} \, dy$$

$$A_m = \frac{2V_0}{a} \int_0^a \sin \frac{m\pi}{b} y \sin^2 \frac{\pi y}{a} \, dy$$

6



Use trivial solution

$$V(\rho, \phi) = \frac{V_0}{\alpha}\phi = \frac{2V_0}{\pi}\phi$$

The field is

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\hat{\rho} - \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{\phi}$$

$$= -\frac{2V_0}{\pi \rho}\hat{\phi}$$