$\underline{Ex}$ Consider two potentials on a spherical surface $V_s = V_0 \cos^2\theta$ and $V_s = V_0 \sqrt{|\cos\theta|}$

Find the potential everywhere.

<u>Inside</u>

$$V(r,\theta) = \sum_n A_n r^n P_n(\cos\theta)$$

Evaluate on surface and use orthogonality

$$\int_{-1}^{1} P_m(\cos\theta) V(a,\theta)\, d(\cos\theta)$$

$$= \sum_n A_n a^n \int_{-1}^{1} P_m(\cos\theta) P_n(\cos\theta)\, d\cos\theta$$

$$= A_m a^m \frac{2}{2m+1}$$

$$A_m = \left[\int_{-1}^{1} P_m(\cos\theta) V(a,\theta)\, d(\cos\theta)\right] \frac{2m+1}{2a^m}$$

Outside

$$V(r,\theta) = \sum_n B_n \, r^{-(n+1)} P_n(\cos\theta)$$

Through similar logic

$$B_m = \frac{(2m+1)\,a^{m+1}}{2} \int_{-1}^{1} P_m(\cos\theta)\, \cancel{P_n(\cos}\, V(a,\theta)\, d\cos\theta$$

Now, work with the potentials given

$$V_s = V_0 \cos^2\theta$$

$$A_n = \frac{2n+1}{2a^n} \int_{-1}^{1} P_n(\cos\theta)\, V_0 \cos^2\theta \, d(\cos\theta)$$

Now what?

Two choices, (1) Evaluate term by term,

$P_0, P_1, P_2 \ldots$

Try to write V_s in terms of a combination of ~~spherical harmonics~~. Legendre polynomials.

$\Rightarrow$ Always try this first.

as $$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$$

$$\cos^2\theta = \frac{2P_2(\cos\theta) + 1}{3}$$

$$= \frac{2}{3}P_2(\cos\theta) + \frac{1}{3}P_0(\cos\theta)$$

since $P_0(\cos\theta) = 1$

$$A_n = \frac{2n+1}{2a^n} V_0 \int_{-1}^{1} \left[\frac{2}{3}P_2(\cos\theta) + \frac{1}{3}P_0(\cos\theta)\right] \cancel{d\cos\theta}\ P_n(\cos\theta)\, d\cos\theta$$

$$A_0 = V_0 \frac{2(0)+1}{2a^0} \cdot \frac{1}{3} \int_{-1}^{1} P_0(\cos\theta)\, P_0(\cos(\theta))\, d\cos\theta$$

$$= \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{2}{2 \cdot 0 + 1} = \frac{1}{3} V_0$$

$$A_2 = V_0 \frac{2 \cdot 2 + 1}{2a^2} \cdot \frac{2}{3} \cdot \int_{-1}^{1} P_2(\cos\theta)\, P_2(\cos\theta)\, d\cos\theta$$

$$= \frac{5}{2a^2} \cdot \frac{2}{3} \cdot \frac{2}{2 \cdot 2 + 1} = \frac{2}{3a^2} V_0$$

All other coefficients zero

$$V_i(r,\theta) = \frac{V_0}{3} P_0(\cos\theta) + \frac{2}{3} \frac{r^2}{a^2} V_0 P_2(\cos\theta)$$

By observation

$$V_{out} = \frac{V_0}{3} \frac{a}{r} P_0 \cos\theta + \frac{2}{3} V_0 \frac{a^3}{r^3} P_2(\cos\theta)$$

- To finish problem compute $\vec{E}$ and σ.

Now try ~~$\sqrt{|\cos\theta|}$~~ $V_s = V_0 \sqrt{|\cos\theta|}$

Unfortunately there is no simple way to write this in terms of $P_i(\cos\theta)$. We are left computing some of the terms.

$$A_n = \frac{2n+1}{2a^n} V_0 \int_{-1}^{1} \sqrt{|\cos\theta|}\, P_n(\cos\theta)\, d\cos\theta$$

$$= \frac{2n+1}{2a^n} V_0 \int_{-1}^{1} \sqrt{|x|}\, P_n(x)\, dx$$

Terms

$$A_0 = \frac{2\cdot 0+1}{2a^0} V_0 \int_{-1}^{1} \sqrt{|x|}\, dx$$

$$= \frac{1}{2} V_0 \left(\frac{|x|^{3/2}}{3/2} \right)_{-1}^{1} = \frac{2}{3} V_0$$

$$A_1 = \frac{2\cdot 1+1}{2a} V_0 \int_{-1}^{1} \sqrt{|x|}\; x\, dx \qquad P_1 = x$$

$$= 0$$

$$A_2 = \frac{2\cdot 2+1}{2a^2} V_0 \int_{-1}^{1} \sqrt{|x|} \cdot \tfrac{1}{2}(3x^2-1)\, dx$$

$$= \frac{5V_0}{2a^2}\left[\int_{-1}^{1} -\frac{\sqrt{|x|}}{2}\, dx + \frac{3}{2}\int_{-1}^{1} x^2\sqrt{|x|}\right]$$

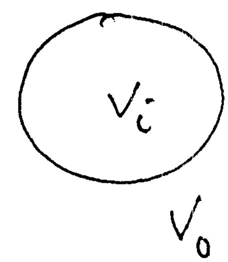
$$= \frac{5V_0}{2a^2}\left[-\frac{x^{3/2}}{3/2}\Big|_0^1 + 3\frac{x^{7/2}}{7/2}\Big|_0^1\right]$$

$$= \frac{5V_0}{2a^2}\left[-\frac{2}{3} + \frac{6}{7}\right]$$

Ex Cylindrical System (infinite) with surface charge density $\sigma(\phi) = \sigma_0 \cos 3\phi$. Compute field.

Sln

$$V_i = \sum_n A_n \rho^n \cos n\phi + B_n \sin n\phi \, \rho^n$$



$$V_o = \sum_n C_n \rho^{-n} \cos n\phi + D_n \rho^{-n} \sin n\phi$$

Boundary Conditions

- $V_i(a,\phi) = V_o(a,\phi)$ Potential continuous
- Field must give correct charge at surface by Gauss.

$$\vec{E}_o \cdot \hat{\rho} A - \vec{E}_i \cdot \hat{\rho} A = \Phi \quad \text{Gauss}$$

$$= \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma(\theta) A}{\varepsilon_0}$$

$$\vec{E}_o \cdot \hat{\rho} = \frac{-\partial V_o}{\partial \rho} \qquad \vec{E}_i \cdot \hat{\rho} = \frac{-\partial V_i}{\partial \rho}$$

$$\Rightarrow \quad \left.\frac{\partial V_i}{\partial \rho}\right|_a - \left.\frac{\partial V_o}{\partial \rho}\right|_a = \frac{\sigma(\phi)}{\varepsilon_o}$$

<u>Apply BC</u>

Continuous $\quad V_i(a,\phi) = V_o(a,\phi)$

$$\sum_n A_n a^n \cos n\phi + B_n a^n \sin n\phi$$

$$= \sum_n C_n a^{-n} \cos n\phi + D_n a^{-n} \sin n\phi$$

<u>By orthogonality</u>

$$A_n a^n = C_n a^{-n} \qquad B_n a^n = D_n a^{-n}$$

Gauss' Law

$$\left.\frac{\partial V_i}{\partial \rho}\right|_a - \left.\frac{\partial V_o}{\partial \rho}\right|_a = \frac{\sigma(\phi)}{\varepsilon_0} = \frac{\sigma_0 \cos 3\phi}{\varepsilon_0}$$

$$\sum_n A_n n a^{n-1} \cos n\phi + B_n n a^{n-1} \sin n\phi$$

$$- \sum_n C_n (-n) a^{-(n+1)} \cos n\phi + D_n (-n) a^{-(n+1)} \sin n\phi$$

$$= \frac{\sigma_0 \cos 3\phi}{\varepsilon_0}$$

By orthogonality

$$B_n n a^{n-1} - \left(-n D_n a^{-(n+1)}\right) = 0$$

$$A_n n a^{n-1} - \left(-n C_n a^{-(n+1)}\right) = 0 \text{ if } n \neq 3$$

$\cancel{A_3 n a^{n-1} = 3a^2 A_3}$ If $n = 3$,

$$A_3 3 a^{3-1} - C_3(-3) a^{-(3+1)} = \frac{\sigma_0}{\varepsilon_0}$$

All terms but A_3, C_3 are solved, if the terms are zero.

$$B_n,\ D_n,\ A_{n\neq 3},\ C_{n\neq 3} = 0$$

$$3a^2 A_3 + 3a^{-4} C_3 = \frac{\sigma_0}{\varepsilon_0}$$

and

$$A_3 a^3 = C_3 a^{-3}$$

$$A_3 = C_3 a^{-6}$$

$$3a^2\left(C_3 a^{-6}\right) + 3a^{-4} C_3 = \frac{\sigma_0}{\varepsilon_0}$$

$$6a^{-4} C_3 = \frac{\sigma_0}{\varepsilon_0}$$

$$C_3 = \frac{\sigma_0 a^4}{6\varepsilon_0}$$

$$A_3 = C_3 a^{-6} = \frac{\sigma_0}{6\varepsilon_0 a^2}$$

$$V_i = \frac{\sigma_0}{6\varepsilon_0 a^2}\rho^3 \cos 3\phi$$

$$V_o = \frac{\sigma_0 a^4}{6\varepsilon_0 \rho^3}\cos 3\phi$$

<u>Field Inside</u>

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\hat{\rho} - \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{\phi} - \frac{\partial V}{\partial z}\hat{z}$$

$$\vec{E}_o = +\frac{\sigma a^4}{2\varepsilon_0 \rho^4}\cos 3\phi\,\hat{\rho} + \frac{\sigma a^4}{2\varepsilon_0 \rho^4}\sin 3\phi\,\hat{\phi}$$

$$\vec{E}_i = -\frac{\sigma}{2\varepsilon_0 a^2}\rho^2 \cos 3\phi\,\hat{\rho} + \frac{\sigma}{2\varepsilon_0 a^2}\rho \sin 3\phi\,\hat{\phi}$$

<u>Work to move + charge from ~~∞ to origin?~~ origin to ∞?</u>

$$W = q\Delta V = qV_i(0) - qV_o(\infty) = 0$$

KE of a charge released at point $\rho = 0$, $\phi = 0$ after long time.

The electric field is radial for $\phi = 0$, so the particle travels in a straight line.

$$\Delta V = V_0(\infty) - V_0(a)$$

$$= 0 - \frac{\sigma_0 a^4}{6 \varepsilon_0 a^3} = \frac{\sigma a}{6 \varepsilon_0}$$

$KE = q \Delta V$ (after long time all ~~KE be~~ PE becomes KE)

$$\frac{1}{2} m v^2 = \frac{q \sigma_0 a}{6 \varepsilon_0}$$

$$v = \sqrt{\frac{2 q \sigma_0 a}{6 \varepsilon_0}}$$