Boundary Conditions and Uniqueness

Suppose we want to find the potential inside some region of space and we know the potential and charge on the boundaries of the region. Can we find more than one solution?

The potential is governed by Poisson's Eqn

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \nabla^2 V = -\rho/\varepsilon_0$$

If there is no charge in the region we have Laplace's eqn.

$$\nabla^2 V = 0$$

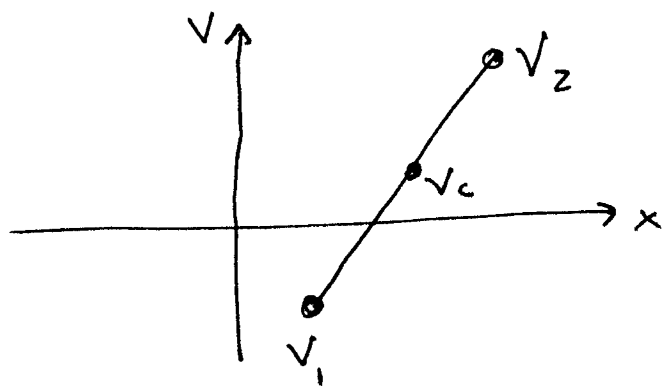
Laplace's eqn has some interesting properties.

Earnshaw's Thm – In a charge free region V does not have a local minimum, except possibly on the boundaries of the region.

$\Rightarrow$ 1D demonstration

$$\nabla^2 V \rightarrow \frac{d^2 V}{dx^2} = 0$$

- Sln $V = ax + b$



- Extrema happen on boundary.
- $\frac{d^2V}{dx^2} \Rightarrow$ no curvature

$\Rightarrow$ $V_c = \frac{V_1 + V_2}{2}$ The potential at the center is the average potential at the boundaries.

$\Rightarrow$ 2D + 3D

The ~~average~~ potential (charge free) at the center of a sphere is the average of the potential over the surface.

$\Rightarrow$ what's happening at the boundaryes is most of the story.

<u>Boundary Conditions</u>

Dirichlet – Specify $V(x)$ on boundary or region

Neumann – Specify $(\nabla \cdot V) \cdot \hat{n} = \sigma$ on the boundary.

<u>Uniqueness Thm I</u> — The solution of Laplace's eqn is uniquely determined if V is specified on the boundary S of some volume.

$\Rightarrow$ The potential V is uniquely determined if V is given on boundary and charge density IN the volume is specified.

<u>Uniqueness Thm II</u> – If a volume V is surrounded by conductors and contains a specified charge density ρ, the electric field in uniquely determined if the total charge on the conductors is given.

<u>Uniqueness Thm III</u> – If the charge density σ is given on the boundary of a region and the charge density is specified inside the region, the potential is uniquely defined up to a constant.

⇒ Beg, borrow, or (usually) steal a solution for the fields. If it matches the boundary conditions you have the right one.

<u>Electrostatic</u>	Magnetostatic
$\nabla \cdot \vec{E} = \rho/\varepsilon_0$	$\nabla \cdot \vec{B} = 0$
$\nabla \times \vec{E} = 0$	$\nabla \times \vec{B} = \mu_0 \vec{J}$
$\nabla \cdot \vec{D} = \rho_f$	$\nabla \cdot \vec{B} = 0$
$\nabla \times \vec{E} = 0$	$\nabla \times \vec{H} = \vec{J}_f$