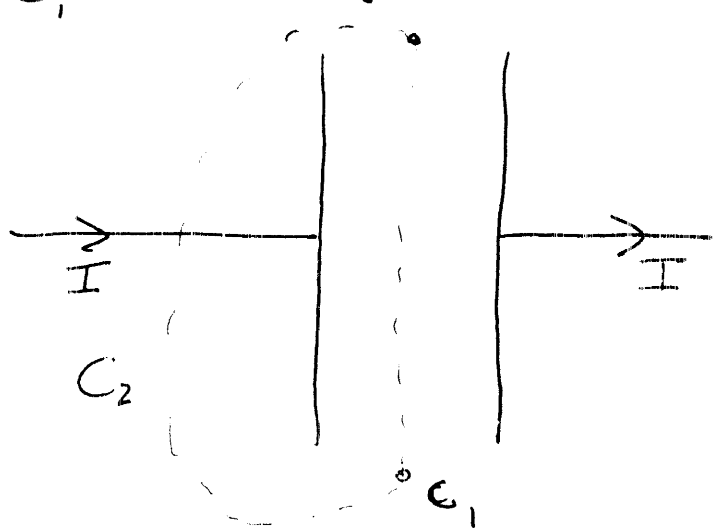
Ampere's Law for Electrodynamics

Something's missing from Ampere's Law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

Consider a charging capacitor and two curves C_1 and C_2



$\oint \vec{B} \cdot d\vec{l}$ must be the same for both curves, but $I_{enc} = 0$ for C_1 and $I_{enc} = I$ for C_2.

In analogy to Faraday's Law, one might consider fixing this inconsistancy by adding a term that involves the time rate of change of electric flux.

The electric flux through the surface bounded C_1 is

$$\Phi_e = EA = \frac{\sigma A}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$$

therefore

$$\frac{d\Phi_e}{dt} = \frac{1}{\varepsilon_0}\frac{dQ}{dt} = \frac{I}{\varepsilon_0}$$

so $\varepsilon_0 \dfrac{d\Phi_e}{dt}$ is zero on surface 2

and I on surface 1.

Therefore, for this limited case

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left(I_{enc} + I_d \right)$$

if $I_d = \varepsilon_0 \dfrac{d\Phi_e}{dt}$ is consistent.

This turns out to be an exact general correction to Ampere's Law where I_d is called the displacement current.

<u>Dfn Displacement Current</u> – A quantity with the dimensions of current required by Ampere's Law.

$$I_d = \varepsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot \hat{n}\, dA$$

<u>Ampere's Law</u>

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 \left(I_{enc} + I_d \right)$$

$$= \mu_0 I_{enc} + \mu_0 \varepsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot \hat{n}\, dA$$

or in differential form

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

We can identify the inconsistency in a different way.

Ampere's Law before the correction is

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

The divergence of a curl is zero

$$\nabla \cdot (\nabla \times \vec{B}) = 0 = \mu_0 \nabla \cdot \vec{J}$$

For magnetostatics $\nabla \cdot \vec{J} = 0$ and we're fine, but in general $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$ so we would be fine if we added $\frac{\partial \rho}{\partial t}$

$$\nabla \cdot (\nabla \times \vec{B}) = 0 = \mu_0 \left(\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} \right)$$

By Gauss' Law,

$$\rho = \varepsilon_0 \nabla \cdot \vec{E}$$

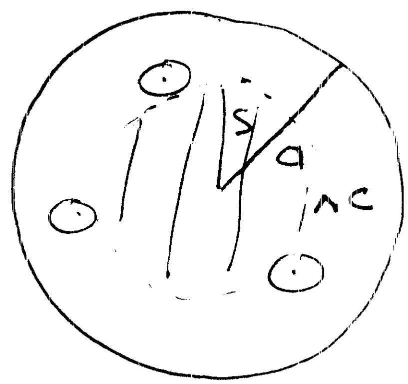
$$\nabla \cdot (\nabla \times \vec{B}) = \mu_0 \left(\nabla \cdot \vec{J} + \varepsilon_0 \nabla \cdot \frac{\partial \vec{E}}{\partial t} \right)$$

This suggests Ampere law should be

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

The displacement current can be used as a source in Ampere's law –

<u>Ex</u> A cylindrical region $s < a$ contains a time varying electric field $\vec{E} = E_0 e^{-t/\tau} \hat{z}$.



The electric flux through a circular surface of radius s is

$$\Phi_e = EA = E\pi s^2$$
$$= E_0 \pi s^2 e^{-t/\tau}$$

The displacement current is then

$$I_d = \varepsilon_0 \frac{d\Phi_e}{dt} = -\frac{\varepsilon_0 \pi s^2}{\tau} e^{-t/\tau}$$

where the – sign indicates I_d flows into page.

We could also define a displacement current density $\vec{J}_d$ where

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \vec{J}_d \right)$$

By observation,

$$\vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

For the system we are considering,

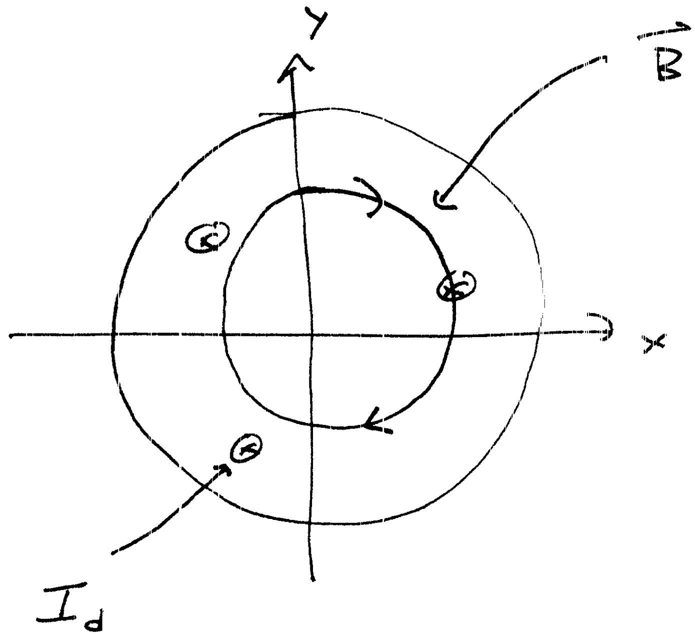
$$\vec{J}_d = -\frac{\varepsilon_0}{\tau} E_0 e^{-t/\tau} \hat{z}$$

and the total displacement current

$$I_d = \int \vec{J}_d \cdot d\vec{a} = J_d A$$

$$= -\frac{\varepsilon_0 \pi s^2 E_0}{\tau} e^{-t/\tau}$$

Since I_d is cylindrically symmetric, the magnetic field is circles about the axis



$$\oint \vec{B} \cdot d\vec{l} = 2\pi s B = \mu_0 \left(\underset{\overset{\|}{0}}{I_{enc}} + I_d \right)$$

$$B = \frac{\mu_0 I_d}{2\pi s} = \left| \frac{\mu_0}{2\pi s} \left(\frac{-\varepsilon_0 E_0 \pi s^2}{\tau} e^{-t/\tau} \right) \right|$$

$$\vec{B} = -\frac{\varepsilon_0 \mu_0 E_0 s}{2\tau} e^{-t/\tau} \hat{\phi}$$