Bessel Functions

Let's relax the restriction that z is uniform.

Consider the systems

Laplace Eqn Cylindrical Coordinates

$$\frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Separated Solution

$$V(\rho, \phi, z) = P(\rho)\,\Phi(\phi)\,Z(z) = R(\rho, \phi)\,Z(z)$$

Separate $Z(z)$ first to give

$$\frac{1}{R}\left(\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 R}{\partial \phi^2}\right) + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0$$

$$Z = \cancel{e^{\pm ikz} \text{ or } \sin kz, \cos kz}$$

$$e^{\pm kx}$$

Now we have

$$\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 R}{\partial \phi^2} + k^2 R = 0$$

$$\rho^2 \frac{\partial^2 R}{\partial \rho^2} + \rho \frac{\partial R}{\partial \rho} + \rho^2 k^2 R + \frac{\partial^2 R}{\partial \phi^2} = 0$$

<u>Seperate Again</u>

$$R = P(\rho)\,\Phi(\phi)$$

$$\underbrace{\frac{1}{P}\left[\rho^2 \frac{\partial^2 P}{\partial \rho^2} + \rho \frac{\partial P}{\partial \rho} + \rho^2 k^2 P\right]}_{+\nu^2} + \underbrace{\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2}}_{-\nu^2} = 0$$

where ν integer

$$\frac{d^2\Phi}{d\phi^2} + \nu^2 \Phi = 0 \qquad \Phi = \begin{matrix} \sin \nu\phi \\ \cos \nu\phi \end{matrix}$$

Leaving

$$\rho^2 \frac{\partial^2 P}{\partial \rho^2} + \rho \frac{\partial P}{\partial \rho} + \rho^2 k^2 P = \nu^2 P$$

$$\frac{\partial^2 P}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial P}{\partial \rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right) P = 0$$

Let $x = k\rho$

$$\frac{d^2P}{dx^2} + \frac{1}{x}\frac{dP}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)P = 0$$

$\Rightarrow$ Bessel's Eqn

<u>Solutions</u> <u>Bessel Functions</u>

~~$J_m(x)$~~ $J_\nu(x)$

<u>Neumann Functions</u>

$N_\nu(x)$

Gamma Function

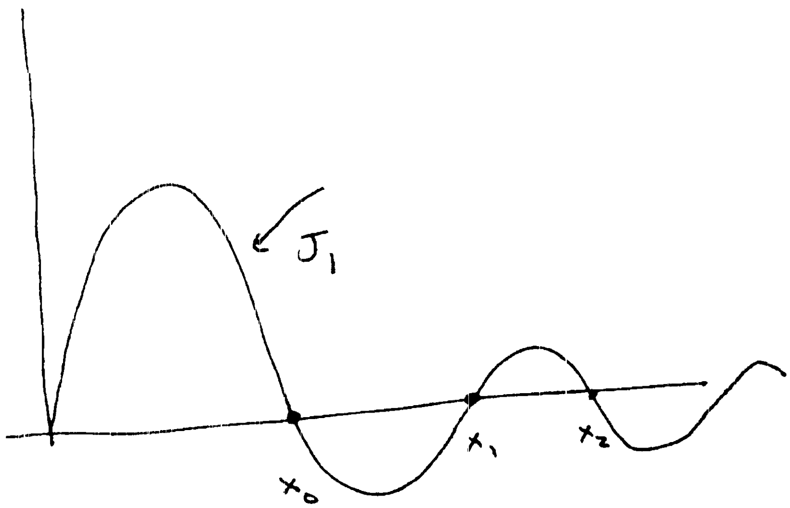
$\Gamma(x) = (x-1)!$

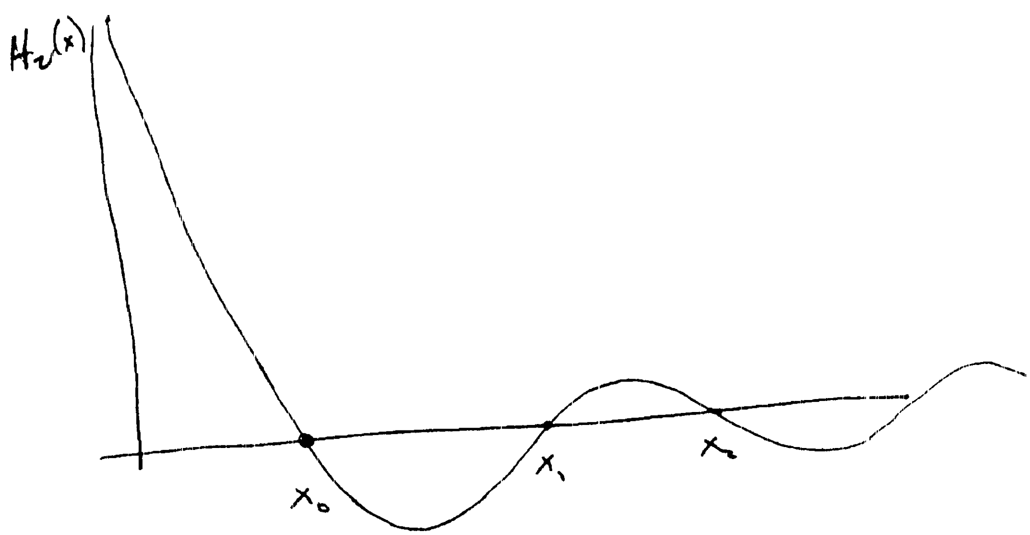
<u>Limits</u>

$$J_\nu(x) = \begin{cases} \dfrac{1}{\Gamma(\nu+1)}\left(\dfrac{x}{2}\right)^{\nu} & x \ll 1 \\ \sqrt{\dfrac{2}{\pi x}}\cos\left(x - \dfrac{\nu\pi}{2} - \dfrac{\pi}{4}\right) & x \gg 1 \end{cases}$$

$$N_\nu(x) = \begin{cases} \dfrac{2}{\pi}\left[\ln\left(\dfrac{x}{2}\right) + 0.5772 + \cdots\right] & \nu = 0 \quad x \ll 1 \\ -\dfrac{\Gamma(\nu)}{\pi}\left(\dfrac{2}{x}\right)^{\nu} & \nu \neq 0 \quad x \ll 1 \\ \sqrt{\dfrac{2}{\pi x}}\sin\left(x - \dfrac{\nu\pi}{2} - \dfrac{\pi}{4}\right) & x \gg 1 \end{cases}$$

$J_\nu(x)$





Each Bessel function has a set of zero

$\nu=0 \quad x_{01} = 2.405 \qquad x_{02} = 5.520 \cdots$

$\nu=1 \quad x_{11} = 3.832 \qquad x_{12} = 7.016 \cdots$

$$J_\nu(x_{\nu n}) = 0$$

Orthogonality

$$\int_0^a \rho\, J_\nu\left(x_{\nu m} \frac{\rho}{a}\right) J_\nu\left(x_{\nu n} \frac{\rho}{a}\right) d\rho$$

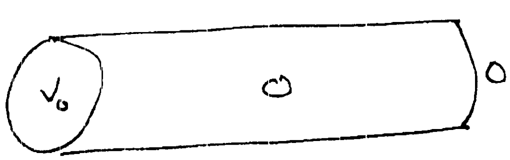
$$= \frac{a^2}{2} \left(J_{\nu+1} (x_{\nu n})\right]^2 \delta_{mn}$$

Completeness

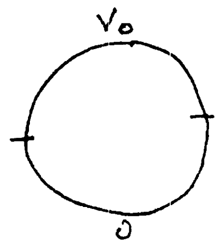
$$f(\rho) = \sum_{n=1}^{\infty} A_{\nu n} J_\nu\left(x_{\nu n} \rho/a\right)$$

$$A_{\nu n} = \frac{2}{a^2 J^2_{\nu+1}(x_{\nu n})} \int_0^a \rho\, f(\rho)\, J_\nu\left(\frac{x_{\nu n} \rho}{a}\right) d\rho$$

So these can be used to solve



The other case we considered is



of length l with both ends zero.

This time the Z separation must yield sines and cosines.

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = -k^2$$

Gives
$$\frac{d^2 R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - \left(1 + \frac{\nu^2}{x^2}\right) R = 0$$

$\Rightarrow$ Solutions Modified Bessel Functions

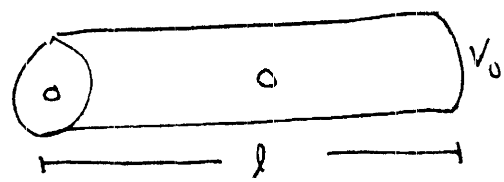
$$I_\nu(x) = i^{-\nu} J_\nu(ix)$$

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix)$$

Hankel Function

$$H_\nu^{(1)}(x) = J_\nu(x) + i N_\nu(x)$$

For our cylinder,



our possible solutions are $1, \ln \rho, \phi, z$

$$\left[J_\nu(\rho k) + N_\nu(\rho k)\right] \times \left[\sin \nu\phi, \cos \nu\phi\right] \left[e^{kz}, e^{-kz}\right]$$

All the trivial solutions can be eliminated N_ν blows up.

To satisfy $V=0$ at $z=0$ take $\dfrac{e^{kz} - e^{-kz}}{2} = \sinh kz$

for the z dependance.

There is no ϕ dependance, so used $\nu = 0$ solution. This

leaves us

$J_0(\rho k) \sinh kz$ for our solutions

Now satisfy the boundary condition $V(a, \phi) = 0$.

$$\Rightarrow \quad J_0(ak) = 0$$

$$\Rightarrow \quad k_{0n} = \frac{x_{0n}}{a}$$ where x_{mn} are the zeros of the Bessel functions.

The complete solution is then

$$V(\rho, \phi, z) = \sum_n A_n J_0(x_{0n} \rho/a) \sinh(k_{0n} z)$$

Satisfy final boundary condition

$$V(\rho, \phi, l) = V_0 = \sum_n A_n J_0(x_{0n} \rho/a) \sinh(k_{0n} l)$$

$$A_n = \frac{2 V_0}{a^2 J_1^2(x_{0n}) \sinh(k_{0n} l)} \int_0^a \rho J_0\left(\frac{x_{0n} \rho}{a}\right) d\rho$$

$$\cancel{= \frac{2V_0}{a^2 J_1^2(x_{0n}) \sinh(k_{0n} l)}}$$

Let $\quad x = \dfrac{x_{on}\,\rho}{a} \qquad dx = \dfrac{x_{on}}{a}\,d\rho$

$$\rho = \frac{a}{x_{on}}$$

$$A_n = \frac{2V_0}{a^2\,J_1^2(x_{on})\,\sinh(k_{on}\ell)}\left(\frac{a}{x_{on}}\right)^2 \int_0^{x_{on}} x\,J_0(x)\,dx$$

$$\int_0^{x_{on}} x\,J_0(x)\,dx = x\,J_1(x)\Big|_0^{x_{on}} = x_{on}\,J_1(x_{on})$$

$$A_n = \frac{2\,V_0\,\cancel{a}}{x_{on}\;J_1(x_{on})\,\sinh(k_{on}\ell)}$$