Magnetostatics - Biot-Savart Law

Magnetostatics - Study of systems of constant current.

$$\Longrightarrow \frac{\partial \rho}{\partial t} = 0$$

$$\Longrightarrow \nabla \cdot \vec{J} = 0$$

<u>Biot-Savart Law</u> - The magnetic field at a point $\vec{r}$ due to a distribution of current $\vec{J}(\vec{r}\,')$ is

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}\,') \times \hat{r}''}{r''^2} \, d\tau'$$

- $\vec{r}\,'' = \vec{r} - \vec{r}\,'$ as before
- Permeability of Free Space

$$\mu_0 = 4\pi \times 10^{-7} \frac{Tm}{A}$$

- Units Tesla $1\,T = \frac{Ns}{Cm} = \frac{N}{Am}$

If the currents are well approximated by lines or sheets.

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_S \frac{\vec{K} \times \hat{r}''}{r''^2} \, da'$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_C \frac{\vec{I} \times \hat{r}''}{r''^2} \, d\ell'$$

$$= \frac{\mu_0}{4\pi} \int_C \frac{I \, d\vec{\ell}' \times \hat{r}''}{r''^2}$$

or very approximately if $v \ll c$.

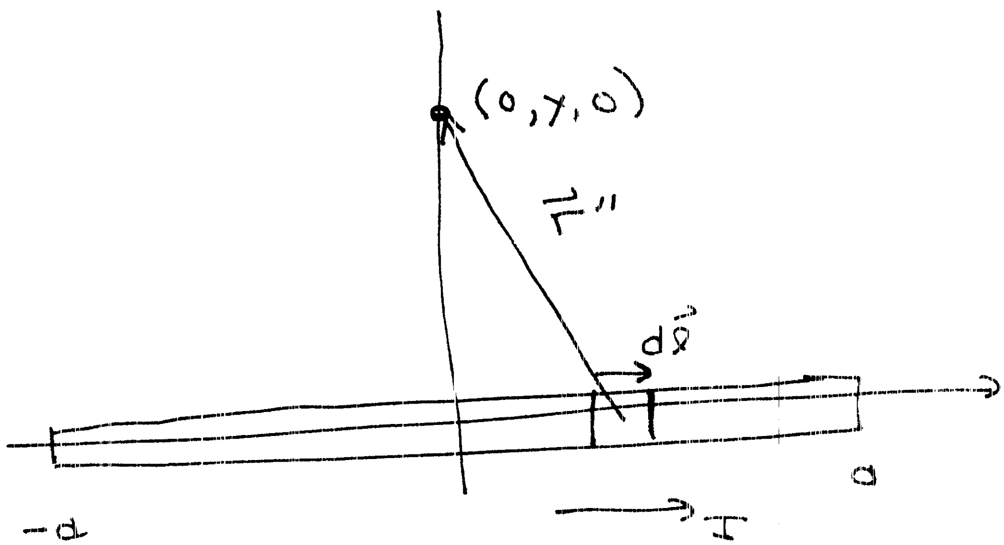
$$\vec{B} = \frac{\mu_0}{4\pi} \frac{q \vec{v} \times \hat{r}''}{r''^2}$$

for a charge q moving at velocity $\vec{v}$.

Ex Compute field of straight wire segment from $-a$ to a along x-axis.

Compute field at $\vec{r} = (0, y, 0)$

Sln



$\vec{r}' = (x', 0, 0)$ $\qquad$ $\vec{r} = (0, y, 0)$

$$\vec{r}'' = \vec{r} - \vec{r}' = (-x', y, 0)$$

Magnetic Field

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_C \frac{d\vec{l}' \times \hat{r}''}{r''^2}$$

or since $\hat{r}'' = \vec{r}''/r''$

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int_c \frac{d\vec{l}' \times \vec{r}''}{r''^3}$$

$$d\vec{l}' = dx'\,\hat{x}$$

$$d\vec{l}' \times \vec{r}'' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx' & 0 & 0 \\ -x' & y & 0 \end{vmatrix}$$

$$= dx'\,y\,\hat{z}$$

$$\vec{B} = \frac{\mu_0 I y \hat{z}}{4\pi}\int_{-a}^{a} \frac{dx'}{\left(\sqrt{x'^2 + y^2}\right)^3}$$

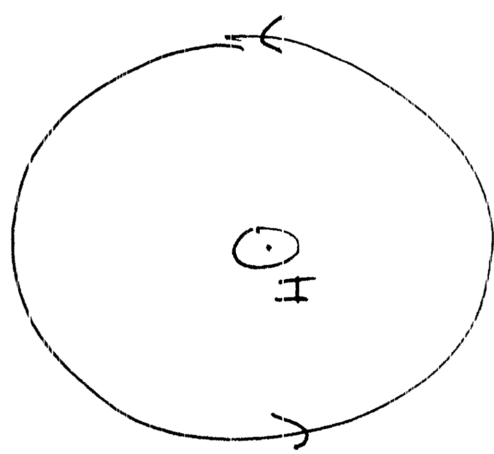
$$\vec{B} = \frac{\mu_0 I y \hat{z}}{4\pi}\left(\frac{2a}{y^2\sqrt{y^2 + a^2}}\right)$$

$$\vec{B} = \frac{\mu_0 I}{2\pi y}\left(\frac{a}{\sqrt{y^2+a^2}}\right)\hat{z}$$

<u>Ex</u> <u>Field of infinite straight wire</u>

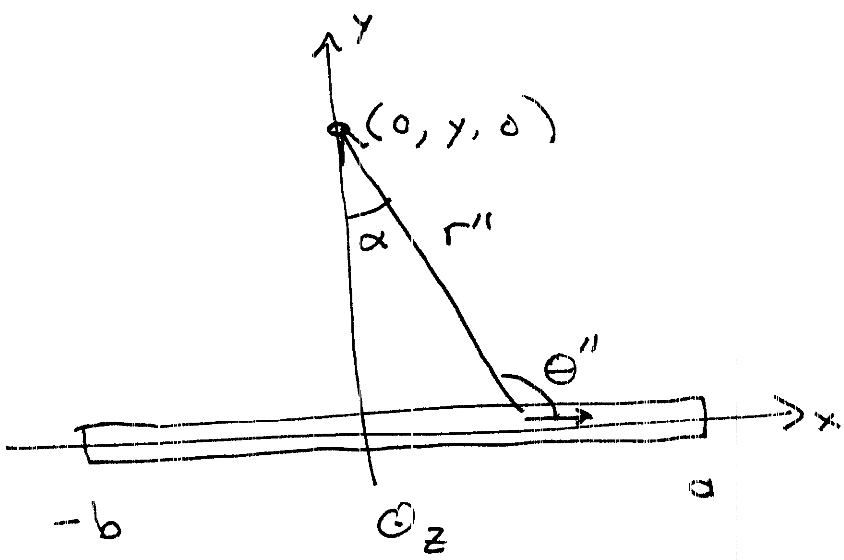
If $a \to \infty$, $\vec{B} = \dfrac{\mu_0 I}{2\pi y}\hat{z}$

or $B = \dfrac{\mu_0 I}{2\pi R}$



<u>RHR Wire</u> - Grab wire with right hand with thumb in direction of current, fingers point in direction of field.

Ex Alternate solution for short wire



Observe, by right-hand rule the field points out of the page.

$$d\vec{B} = |d\vec{B}|\,\hat{z}$$

$$|d\vec{B}| = \frac{\mu_0 I}{4\pi} \frac{|d\vec{l}'||\hat{r}''|}{r''^2} \sin\theta''$$

$$= \frac{\mu_0 I}{4\pi} \frac{\sin\theta''\, dx'}{r''^2}$$

where θ'' is the angle between $d\vec{l}'$ and $\hat{r}''$

$$\theta'' = \alpha + \pi/2$$

$$\cos\alpha = \frac{y}{r''} = \cos(\theta'' - \pi/2)$$

$$= \sin\theta''$$

$$r'' = \frac{y}{\sin\theta''}$$

$$|dB| = \frac{\mu_0 I}{4\pi} \frac{\sin\theta''\, dx'}{(y/\sin\theta'')^2}$$

$$= \frac{\mu_0 I}{4\pi y^2} \sin^3\theta''\, dx'$$

$$\sin\alpha = \frac{x'}{r''} = \sin(\theta'' - \pi/2)$$

$$= -\cos(\theta'')$$

$$x' = -r''\cos\theta'' = -y\frac{\cos\theta''}{\sin\theta''} = -y\cot\theta''$$

$$dx' = -y \, d\cot\theta'' = y(1+\cot^2\theta'')\, d\theta''$$

$$= y\csc^2\theta'' d\theta'' = y\,\frac{d\theta''}{\sin^2\theta''}$$

$$|dB| = \left| \frac{\mu_0 I}{4\pi y} \sin\theta'' d\theta'' \right|$$

$$\vec{B} = \left| \frac{\mu_0 I}{4\pi y} \hat{z} \int_{\theta_b''}^{\theta_a''} \sin\theta'' d\theta'' \right|$$

$$= \left| \frac{-\mu_0 I}{4\pi y} \hat{z} \left(\cos\theta_a'' - \cos\theta_b'' \right) \right|$$

$$= \left| \frac{-\mu_0 I}{4\pi y} \hat{z} \left(\cos\left(\alpha_a - \frac{\pi}{2}\right) - \cos\left(\alpha_b - \pi/b\right) \right) \right|$$

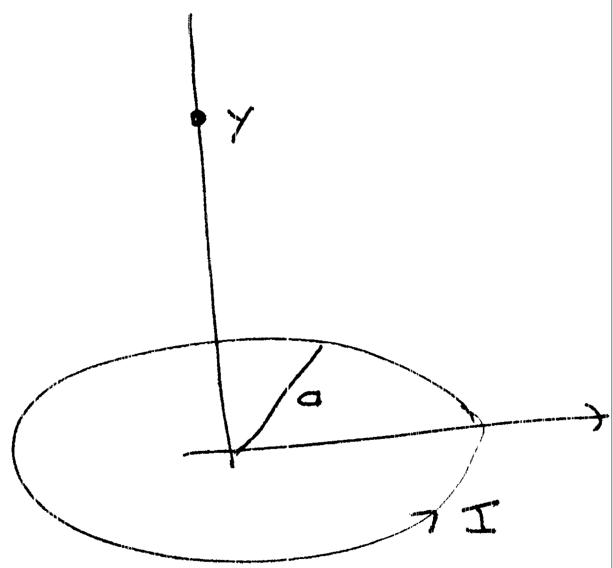
$$= \left| \frac{-\mu_0 I}{4\pi y} \hat{z} \left(\sin\alpha_a - \sin\alpha_b \right) \right|$$

$$\vec{B} = \frac{\mu_0 I}{4\pi y} \hat{z} \left(\sin\alpha_a - \sin\alpha_b \right)$$

As $\alpha_a, \alpha_b \rightarrow \frac{\pi}{2}, -\pi/2$

$$\vec{B} \rightarrow \frac{\mu_0 I}{2\pi y}\hat{z} \quad \checkmark$$

<u>Ex</u> ~~Disk~~ Ring of Current in x-y plane



$$\vec{I} = I\hat{\phi}$$

$$\vec{B} = \cancel{\frac{\mu_0}{4\pi}\int_c \frac{\vec{I}\times\hat{r}''}{r''^2}} = \frac{\mu_0}{4\pi}\int \frac{I\,d\vec{l}'\times\hat{r}''}{r''^2}$$

$$d\vec{l}' = a\,d\phi'\,\hat{\phi}$$

$$\vec{r}' = (a\cos\phi',\ a\sin\phi',\ 0) = a\hat{s}$$

$$\vec{r} = (0,\ 0,\ z)$$

$$\vec{r}'' = \vec{r} - \vec{r}' = (-a\cos\phi',\ -a\sin\phi,\ z)$$

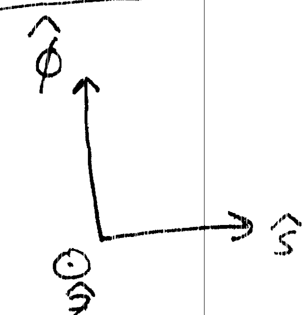
$$= -a\hat{s} + z\hat{z}$$

$$d\vec{l}' \times \vec{r}'' = (a\,d\phi'\,\hat{\phi}') \times (-a\hat{s}' + z\hat{z})$$

$$= -a^2\,d\phi'(\hat{\phi}' \times \hat{s}') + az\,d\phi'\,\hat{\phi}' \times \hat{z}$$

Cylindrical Right-handed Triple

$$\hat{s} \times \hat{\phi} = \hat{z}$$



$$\hat{\phi} \times \hat{s} = -\hat{z}$$

$$\hat{\phi} \times \hat{z} = \hat{s}$$

$$d\vec{l}' \times \vec{r}'' = a^2 \hat{z}\, d\phi' + a z\, d\phi' \hat{s}$$

$$\vec{B} = \underbrace{\frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{d\phi' \hat{s}' a z}{(z^2+a^2)^{3/2}}} + \underbrace{\frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{d\phi' a^2 \hat{z}}{(z^2+a^2)^{3/2}}}$$

$$\underbrace{\frac{\mu_0 I a z}{(z^2+a^2)^{3/2}} \int_0^{2\pi} d\phi' \hat{s}'} \qquad \frac{\mu_0 I a^2 \hat{z}}{2(z^2+a^2)^{3/2}}$$

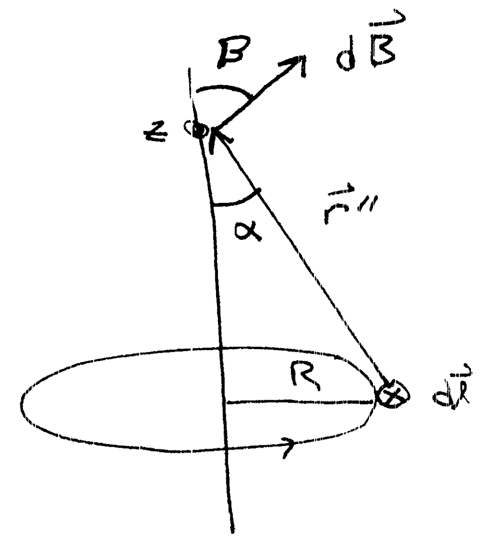
$$\underbrace{\frac{\mu_0 I a z}{(z^2+a^2)^{3/2}} \int_0^{2\pi} d\phi' (\cos\phi', \sin\phi', 0)}$$

$$0$$

$$\vec{B} = \left(\frac{\mu_0 I}{2}\right) \frac{a^2}{(z^2+a^2)^{3/2}}$$

At $z=0$, $$\vec{B} = \frac{\mu_0 I}{2a} \hat{z}$$

That was harder than it needed to be –



$\alpha + B + \pi/2 = \pi$

$\sin(\alpha) = \dfrac{R}{r''}$

$B = \pi/2 - \alpha$

By symmetry, only the z-component of the magnetic field survives.

$$|dB_z| = \frac{\mu_0 I}{4\pi} \frac{d\ell}{r''^2} \cdot \cos B$$

$$= \frac{\mu_0 I}{4\pi} \frac{d\ell}{r''^2} \sin\alpha$$

There is no $\sin\theta$ coming from cross-product because $d\vec{\ell} \perp \hat{r}''$ ~~sin~~ giving $\sin\frac{\pi}{2} = 1$

$$\vec{B} = \hat{z} \int |dB_z|$$

$$= \frac{\mu_0 I}{4\pi} \frac{\sin\alpha}{r''^2} \int_0^{2\pi} dl \ \hat{z}$$

$dl = R\,d\phi$

$$= \frac{\mu_0 I}{2} \frac{R \sin\alpha}{r''^2} \hat{z}$$

$$= \left(\frac{\mu_0 I}{2}\right) \frac{R}{(R^2+z^2)} \cdot \frac{R}{\sqrt{R^2+z^2}} \hat{z}$$

$$= \frac{\mu_0 I}{2} \frac{R^2}{(R^2+z^2)^{3/2}}$$