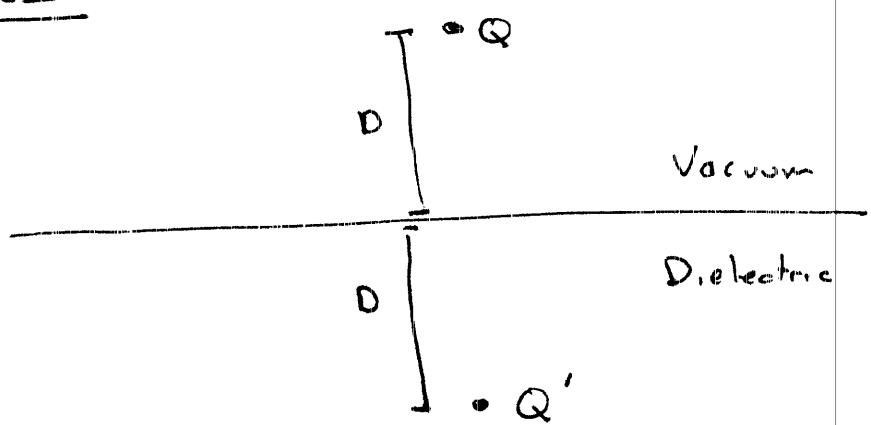
Boundary Value Problems + Dielectrics

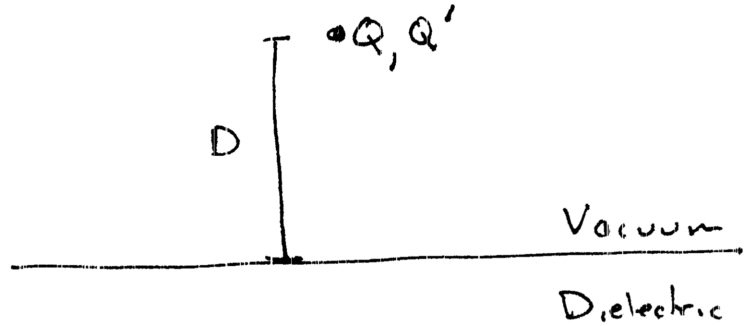
Image Solution – There is no good qualitative way to recognize the solution, just write the surface charge density and map it on the conductor image solution.

Case I – Field outside of dielectric



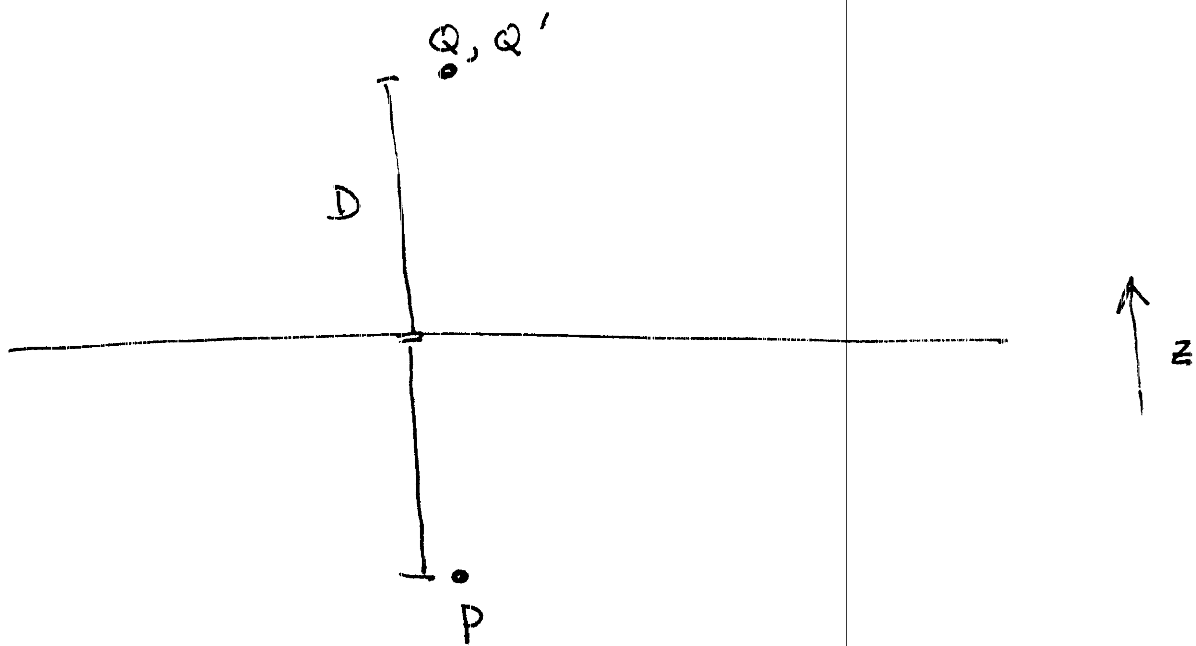
$$Q' = -\frac{\varepsilon_r - 1}{\varepsilon_r + 1} Q$$

Case II **Field inside dielectric**



$$Q' = -\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1}\right) Q$$

Ex A point charge Q is a distance D from a planar dielectric with $\varepsilon_r = 2$. Compute field a distance D inside the dielectric.



$$Q' = -\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1}\right) Q = -\left(\frac{2-1}{2+1}\right) = -\frac{2}{3} Q$$

Use Coulomb's Law to get the field,

$$\vec{E}_P = \frac{k(Q+Q')}{(2D)^2}(-\hat{z})$$

$$= \cancel{k(2/3Q)} \quad \frac{-k(\frac{1}{3}Q)}{(2D)^2} = \frac{-kQ}{12D^2}$$

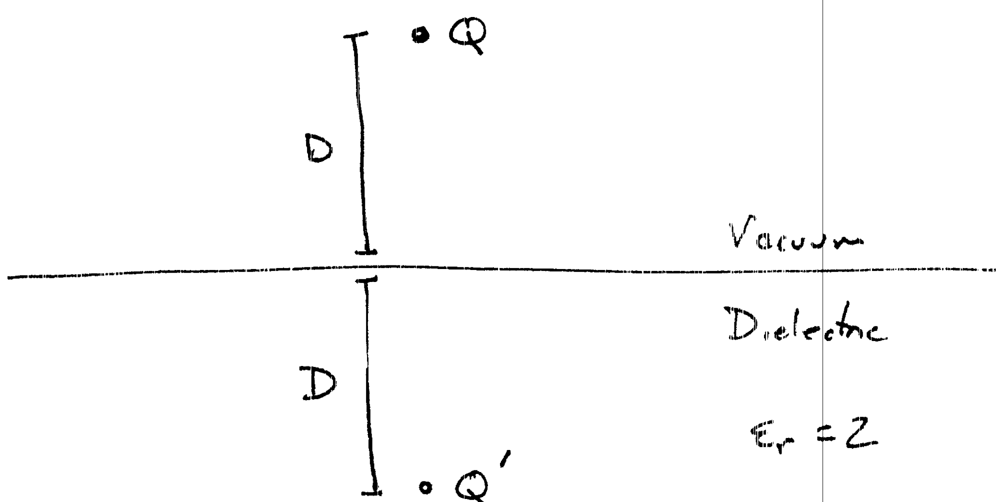
$\Rightarrow$ Dielectric reduces field by $\frac{1}{3}$ not $\frac{1}{2}$

Polarization

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 (\varepsilon_r - 1) \vec{E} = \varepsilon_0 \vec{E}$$

$$= -\frac{\varepsilon_0 k Q}{12 D^2} \hat{z} = -\frac{Q}{48\pi D^2} \hat{z}$$

Ex Force on Point Charge Due to Dielectric Plane



$$Q' = \frac{-(\varepsilon_r - 1)}{(\varepsilon_r + 1)} Q = -\frac{2}{3} Q$$

$$\vec{F}_Q = \frac{k Q Q'}{(2D)^2} \hat{z} = \frac{k (Q)(-\frac{2}{3} Q)}{4 D^2} \hat{z}$$

$$= -\frac{k Q^2}{6 D^2} \hat{z}$$

Boundary Value Problems -

Boundary Conditions

Continuous $\quad V_i(a,\theta) = V_o(a,\theta)$

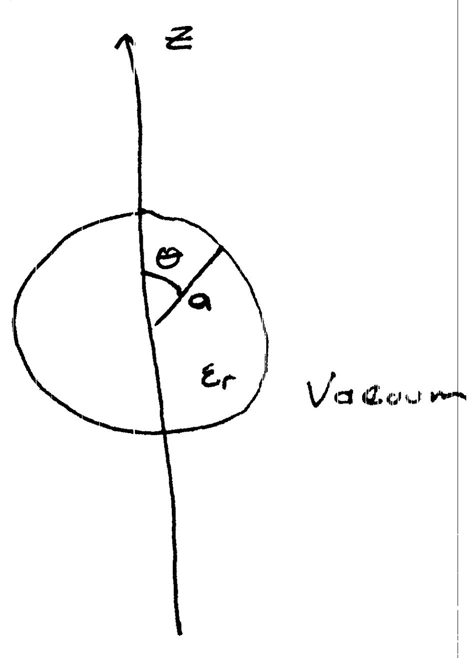
$\vec{D}$ field discontinuous by σ_f

$$\varepsilon_{out} \frac{\partial V_o}{\partial r}\Big|_a - \varepsilon_{r,in} \frac{\partial V_i}{\partial r}\Big|_a = -\sigma_f$$

Ex From book, dielectric sphere in uniform external field $\vec{E} = E_0 \hat{z}$

$\Rightarrow \quad V_o \longrightarrow -E_0 z \quad$ as $\quad r \to \infty, \quad \theta = 0$

No free charge anywhere, but still must impose electrostatic B.C. at surface of sphere.



Only actual charge is bound charge at surface, so the potential satisfiest Laplace's eqn inside and outside sphere.

$$V_i = \sum r^n P_n(\cos\theta) A_n$$

$$V_o = \sum B_n r^{-(n+1)} P_n(\cos\theta) \quad - E_o \underbrace{r\cos\theta}_{z}$$

explosive term to satisfy BC at ∞

<u>V is continuous</u>

$$V_i(a,\theta) = V_o(a,\theta)$$

$$\sum_n A_n a^n P_n(\cos\theta) = \sum B_n a^{-(n+1)} P_n(\cos\theta) - E_o \underbrace{a\cos\theta}_{P_1}$$

<u>Term by term</u>

<u>$n \neq 1$</u> $\quad A_n a^n = B_n a^{-(n+1)}$

$$B_n = A_n a^{2n+1}$$

<u>$n = 1$</u> $\quad A_1 a = \dfrac{B_1}{a^2} - E_o a$

Electrostatic B.C.

$$\varepsilon_{r,out} \frac{\partial V_o}{\partial r}\bigg|_a - \varepsilon_{r,in} \frac{\partial V_i}{\partial r}\bigg|_a = -\sigma_f$$

$$\varepsilon_{r,out} = 1 \qquad \sigma_f = 0$$

$$\frac{\partial V_o}{\partial r}\bigg|_a = \varepsilon_r \frac{\partial V_i}{\partial r}\bigg|_a$$

$$\frac{\partial V_o}{\partial r}\bigg|_a = \sum_n -(n+1) B_n a^{-(n+2)} P_n(\cos\theta) \quad - E_0 P_1$$

$$\frac{\partial V_i}{\partial r}\bigg|_a = \sum_n A_n n a^{n-1} P_n(\cos\theta)$$

$$\frac{\partial V_o}{\partial r}\bigg|_a - \varepsilon_r \frac{\partial V_i}{\partial r}\bigg|_a = \sum_n P_n(\cos\theta)\left[-(n+1) B_n a^{-(n+2)} - \varepsilon_r A_n n a^{n-1}\right]$$

$$- E_0 P_1$$

$$= \sum_n P_n(\cos\theta)\left[-(n+1) a^{n-1} - \varepsilon_r n a^{n-1}\right] A_n$$

$$- E_0 P_1$$

for $n \neq 1$

By orthogonality, $A_n \quad n \neq 1 \quad = 0$

For $n = 1$,

$$-(1+1) B_1 a^{-3} - \varepsilon_r A_1 (1) a^0 - E_0 = 0$$

and from earlier

$$A_1 a = \frac{B_1}{a^2} - E_0 a$$

$$A_1 = \frac{B_1}{a^3} - E_0$$

$$-\frac{2B_1}{a^3} - \varepsilon_r \left(\frac{B_1}{a^3} - E_0 \right) - E_0 = 0$$

$$-\frac{B_1}{a^3} (\varepsilon_r + 2) + \varepsilon_r E_0 - E_0 = 0$$

$$B_1 = a_3 E_0 \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right)$$

$$A_1 = \frac{B_1}{a^3} - E_0 = E_0 \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2} - 1 \right)$$

$$= \frac{-3}{\varepsilon_r + 2} E_0$$

So the full potential,

Inside

$$V_i = \cancel{A_1} \; A_1 r P_1(\cos\theta) = A_1 r \cos\theta$$

$$= A_1 z = -\frac{3z}{\varepsilon_r + 2} E_0$$

Field

$$\vec{E}_i = -\nabla V = \frac{3}{\varepsilon_r + 2} E_0 \hat{z}$$

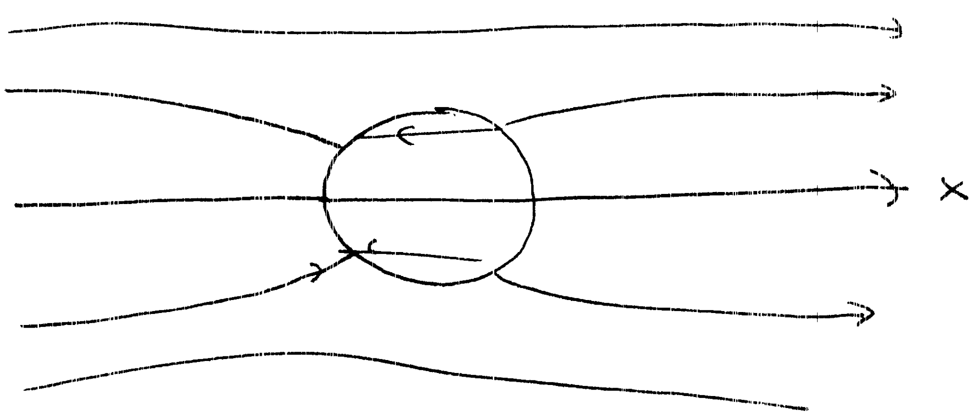
Outside

$$V_o = B_1 r^{-2} P_1(\cos\theta) - E_0 r \cos\theta$$

$$= \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) \frac{a^3}{r^2} E_0 \cos\theta - E_0 z$$

Note -- Dipole field.

Ex Cylinder in External Field



As $s \to \infty$, $V \to -E_0 x = -E_0 s \cos\phi$

~~Field In~~ Potential Inside

$$V_i = \sum_n A_n s^n \cos n\phi + \underbrace{B_n s^n \sin n\phi}_{\text{Discard by symmetry}}$$

$$V_o = \sum_n C_n s^{-n} \cos n\phi + \overbrace{D_n s^{-n} \sin n\phi} - E_0 s \cos\phi$$

Again, no free charge so

$$\left.\frac{\partial V_o}{\partial s}\right|_a - \epsilon_r \left.\frac{\partial V_i}{\partial s}\right|_a = 0$$

All terms are zero except $n=2$

For $n=1$, V continuous

$$V_i(a,\phi) = A_1 a \cos\phi = V_o(a,\phi) = \frac{C_1}{a}\cos\phi - E_o a\cos\phi$$

$$A_1 = \frac{C_1}{a^2} - E_o$$

$$\left.\frac{\partial V_o}{\partial s}\right|_a = \varepsilon_r \left.\frac{\partial V_i}{\partial s}\right|_a$$

$$A_1 \cos\phi = \varepsilon_r\left(-\frac{C_1}{a^2}\cos\phi - E_o\cos\phi\right)$$

$$A_1 = -\varepsilon_r\left(\frac{C_1}{a^2} + E_o\right) = \frac{C_1}{a^2} - E_o$$

$$E_o(1-\varepsilon_r) = \frac{C_1}{a^2}(1+\varepsilon_r)$$

$$C_1 = a^2 E_o\left(\frac{1-\varepsilon_r}{1+\varepsilon_r}\right)$$

$$A_1 = \frac{C_1}{a^2} - E_o = E_o\left(\frac{1-\varepsilon_r}{1+\varepsilon_r} - 1\right)$$

$$= E_o\left(\frac{-2\varepsilon_r}{1+\varepsilon_r}\right)$$

Potential Inside

$$V_i = -2E_0 \frac{\varepsilon_r}{\varepsilon_r + 1} s \cos\phi$$

$$= -2E_0 \frac{\varepsilon_r}{\varepsilon_r + 1} x$$

Field Inside

$$\vec{E}_i = -\nabla V = E_0 \frac{\varepsilon_r}{\varepsilon_r + 1} \hat{x}$$

Potential Outside

$$V_o = -\frac{(\varepsilon_r - 1)}{(\varepsilon_r + 1)} \frac{E_0}{s} \cos\phi \quad - E_0 s \cos\phi$$